Graphing Using Transformations

Horizontal and Vertical Shifts

- **Transformations:** techniques that help sketch graphs with simple modifications of common functions

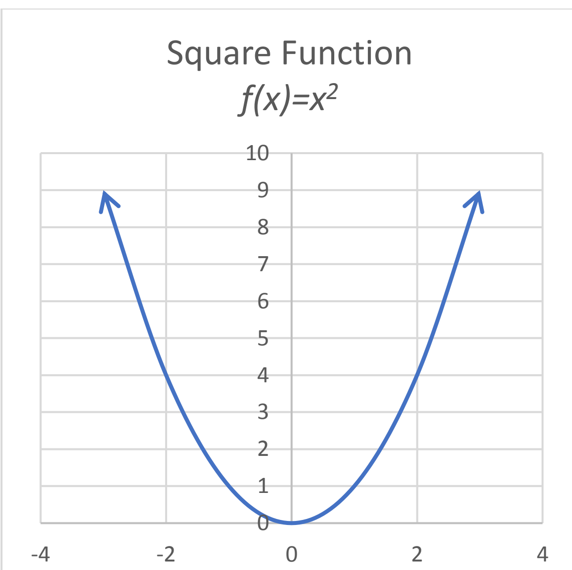
Vertical Shifts	
To Graph	Shift the graph of f(x)
f(x) + c	c units upward
f(x) – c	c units downward

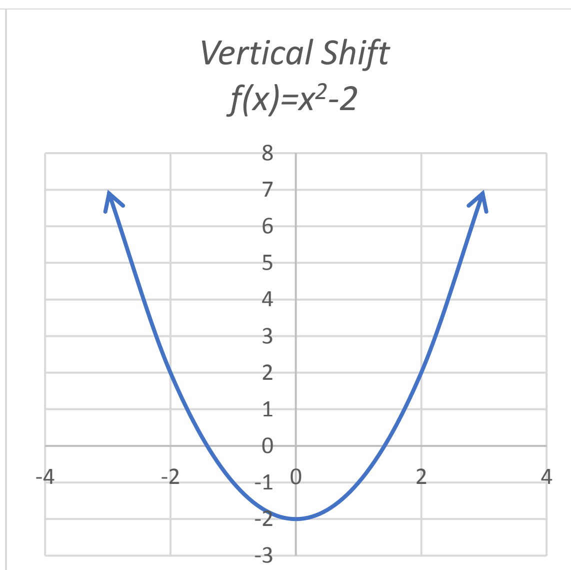
Horizontal Shifts	
To Graph	Shift the graph of f(x)
f(x + c)	c units to the left
f(x – c)	c units to the right

- **Example 1:** Sketch the graph of the given function using horizontal and vertical shifts:

$$g(x) = x^2 - 2$$

In this case, the function to start with is $f(x) = x^2$.
Since $g(x) = x^2 - 2 = f(x) - 2$, the function has a shift outside the function which means it is a vertical shift. Since the two is negative, the shift is two units downward.

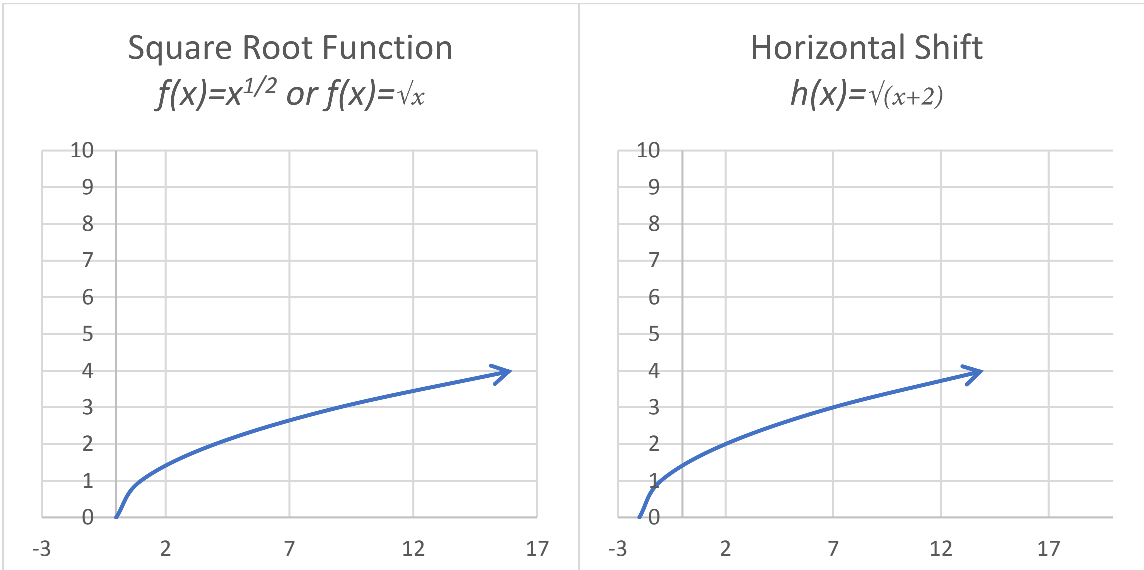




- **Example 2:** Sketch the graph of the given function using horizontal and vertical shifts and find the domain and range of the function:

$$h(x) = \sqrt{x+2}$$

In this case, the function to start with is $f(x) = \sqrt{x}$.
Since $h(x) = \sqrt{x+2} = f(x+2)$, the function has a shift inside the function which means it is a horizontal shift. Since the two is positive, the shift is two units to the left.

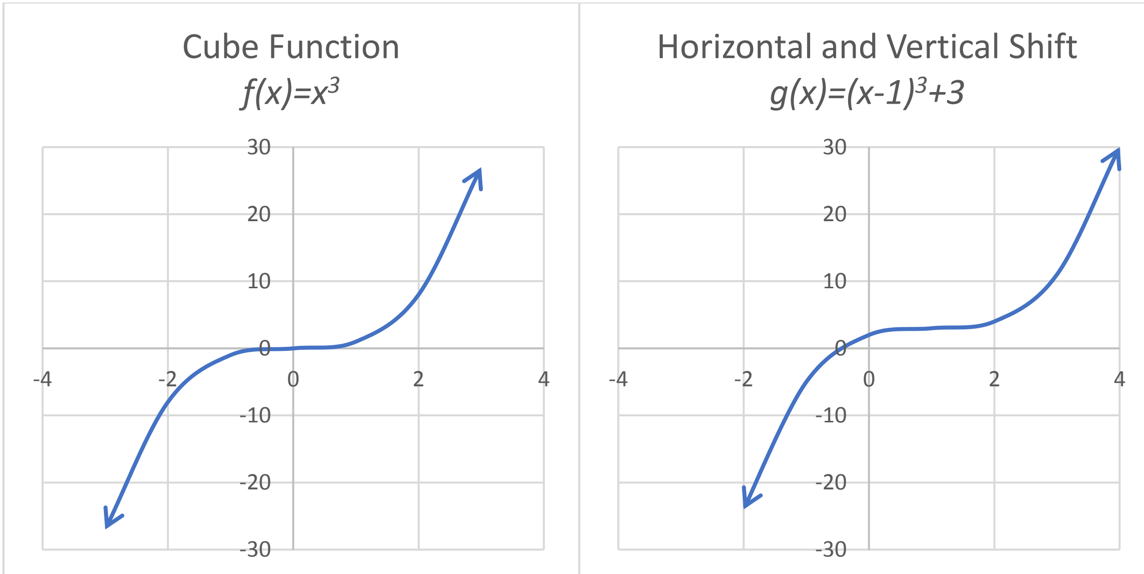


The domain of the function is $[-2, \infty)$, and the range is $[0, \infty)$.

- **Example 3:** Sketch the graph of the given function using horizontal and vertical shifts and find the domain and range of the function:

$$g(x) = (x-1)^3 + 3$$

In this case, the function to start with is $f(x) = x^3$
Since $g(x) = f(x-1) + 3$, the function has a shift inside the function which means it has a horizontal shift (1 unit to the right because it is negative). It also has a shift outside the function which means it is a vertical shift (3 units upward because it is positive).



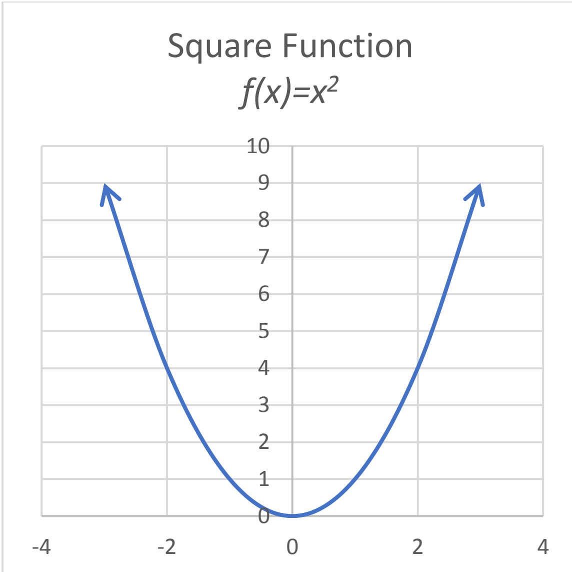
The domain of the function is $(-\infty, \infty)$, and the range is $(-\infty, \infty)$.

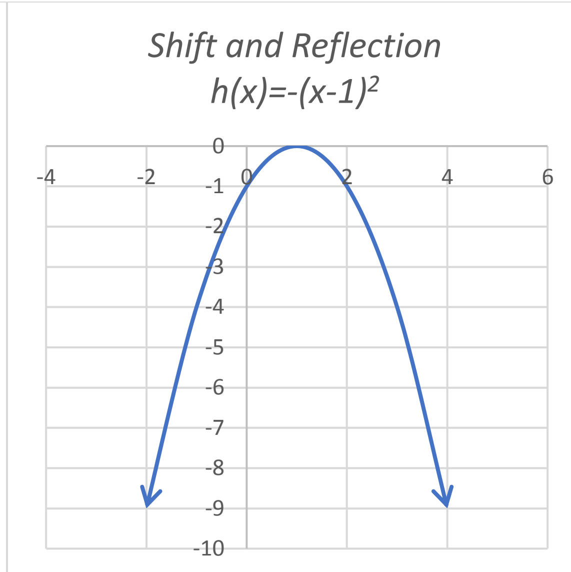
Reflection about the Axes

- The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ about the x-axis.
- The graph of $f(-x)$ is obtained by reflecting the graph of $f(x)$ about the y-axis.
- **Example 1:** Sketch the graph of the given function using reflections and shifts:

$$h(x) = -(x-1)^2$$

In this case, the function to start with is $f(x) = x^2$.
Since $h(x) = -f(x-1)$, the function has a shift inside the function which means it is a horizontal shift. Since the one is negative, the shift is one unit to the right.
The function also has a negative outside the function which means the function is reflected about the x-axis.



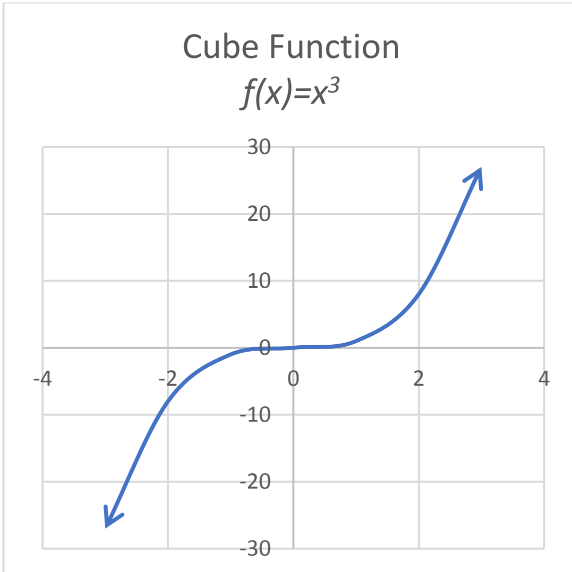


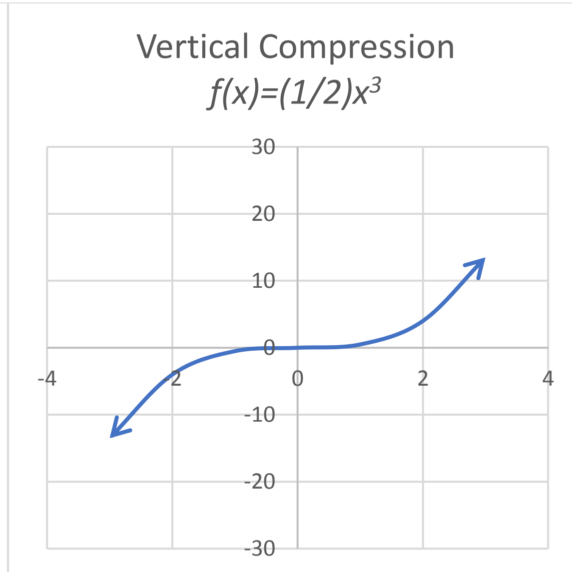
Stretching and Compressing

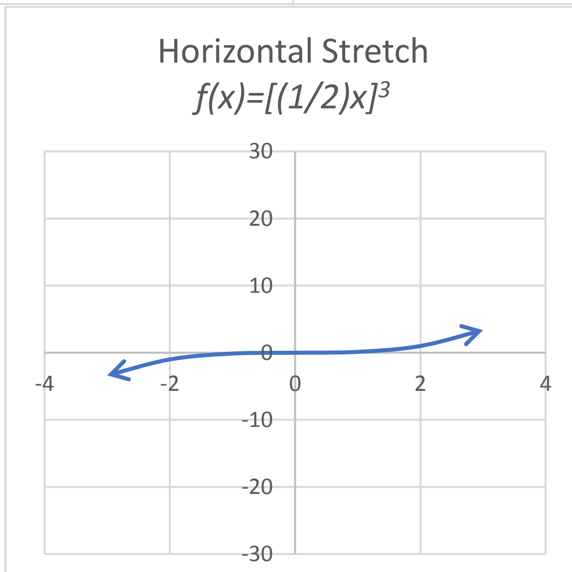
Vertical Stretching and Compressing *cf(x)*	
Vertically stretching	If c > 1
Vertically compressing	If 0 < c < 1

Horizontal Stretching and Compressing *f(cx)*	
Horizontally stretching	If 0 < c < 1
Horizontally compressing	If c > 1

- **Example 1:** Given the graph of $f(x)$, graph: $\frac{1}{2}f(x)$ and $f\left(\frac{1}{2}x\right)$.







To sketch a graph with multiple transformations, follow an "inside out" approach to determine the order of transformations, and just do one of the transformations at a time.

Operations on Functions and Composition of Functions

Operations on Functions

- Consider two functions $f(x) = x^2 + 4x + 3$ and $g(x) = x - 4$
- **Addition:** $f(x) + g(x) = x^2 + 4x + 3 + x - 4 = x^2 + 5x - 1$ (the **sum function**)
- **Subtraction:** $f(x) - g(x) = x^2 + 4x + 3 - (x - 4) = x^2 + 3x + 7$ (the **difference function**)

- **Multiplication:** $f(x) * g(x) = (x^2 + 4x + 3)(x - 4) = x^3 - 13x - 12$ (the **product function**)
- **Division:** $\frac{f(x)}{g(x)} = \frac{(x^2+4x+3)}{(x-4)}$ (the **quotient function**)
- The **domain of the sum function, difference function, and product function** is the intersection of the domains of the two functions (any number that is in both domains is in this new domain).
- The **domain of the quotient function** is the intersection of the domains of the two functions except any values that make the denominator zero must also be eliminated.
- **Example 1:** For functions $f(x) = \sqrt{3 - x}$ and $g(x) = \sqrt{x - 2}$, determine the sum function, difference function, product function, and quotient function. State the domain of these four new functions.

  Sum function: $f(x) + g(x) = \sqrt{3 - x} + \sqrt{x - 2}$

  Difference function: $f(x) - g(x) = \sqrt{3 - x} - \sqrt{x - 2}$

  Product function: $f(x) * g(x) = \sqrt{3 - x} * \sqrt{x - 2} = \sqrt{(3 - x)(x - 2)} = \sqrt{-x^2 + x - 6}$

  Quotient function: $\frac{f(x)}{g(x)} = \frac{\sqrt{3-x}}{\sqrt{x-2}} = \sqrt{\frac{3-x}{x-2}}$

  The domain of $f(x) = \sqrt{3 - x}$ is $(-\infty, 3]$.

  The range of $g(x) = \sqrt{x - 2}$ is $[2, \infty)$.

  The domain of the sum, difference, and product functions is

  $$(-\infty, 3] \cap [2, \infty) = [2,3]$$

  The quotient function has the additional constraint that the denominator cannot be equal to 0. This implies that $x \neq 2$, so the domain of the quotient function is $(2,3]$.

Composition of Functions

Composition of Functions			
Notation	Words	Definition	Domain
$f \circ g$	*f* composed with *g*	$f(g(x))$	The set of all real numbers x in the domain of *g* such that *g(x)* is also in the domain of *f*.
$g \circ f$	*g* composed with *f*	$g(f(x))$	The set of all real numbers x in the domain of *f* such that *f(x)* is also in the domain of *g*.

- It is important not to confuse composition of functions with the multiplication sign:

$$(f \cdot g)(x) = f(x)g(x)$$

- **Example 1:** Given the functions $f(x) = x^2 - 2$ and $g(x) = x + 3$, find $(f \circ g)(x)$.
  Write $f(x) = x^2 - 2$
  Express the composite function: $f(g(x)) = (g(x))^2 - 2$
  Substitute $g(x) = x + 3$ into *f*: $f(g(x)) = (x + 3)^2 - 2$
  Eliminate parenthesis: $f(g(x)) = x^2 + 6x - 1$
  $$(f \circ g)(x) = f(g(x)) = x^2 + 6x - 1$$

- **Example 2:** Given the functions $f(x) = \frac{1}{x-1}$ and $g(x) = \sqrt{x+4}$, find the domain of $f(g(x))$. Do not find the composite function.
  Find the domain and range of $g(x)$ first because it is the inside function.
  Domain of $g(x)$: $[-4, \infty)$
  Range of $g(x)$: $[0, \infty)$
  Now, the domain of *f* is the set of all numbers except 1. We need to eliminate any values of *x* in the domain of *g* that correspond to $g(x) = 1$.
  $$\sqrt{x+4} = 1$$
  $$x + 4 = 1$$
  $$x = -3$$
  We must eliminate $x = -3$ from the domain of *g(x)*.
  The domain of $f(g(x))$ is $[-4, -3) \cup (-3, \infty)$.

- **Example 3:** Given the functions $f(x) = x^2 + 3$ and $g(x) = 10 - x^2$, evaluate $f(g(3))$ and $f(g(-4))$.
  $$f(g(3))$$
  $$g(3) = 10 - (3)^2 = 1$$
  $$f\big(g(3)\big) = f(1)$$
  $$f(1) = (1)^2 + 3 = 4$$
  $$f\big(g(3)\big) = 4$$

  $$f(g(-4))$$
  $$g(-4) = 10 - (-4)^2 = -6$$
  $$f\big(g(-4)\big) = f(-6)$$
  $$f(-6) = (-6)^2 + 3 = 39$$
  $$f\big(g(-4)\big) = 39$$