Systems of Linear Equations in Two Variables

Types of Solutions

- When we say that we are going to solve a system of equations, it means that we are going to find numerical values for all the unknown variables that satisfy the different equations we are given. For example, notice that the solution $x = 5$ and $y = 4$ solves the system

$$3x + 4y = 31$$
$$2x + 3y = 22$$

because

$$3(5) + 4(4) = 15 + 16 = 31$$
$$2(5) + 3(4) = 10 + 12 = 22$$

- **Independent Systems:**
    - A system is said to be independent if each there is only one solution for the system. This means that each unknown has a unique value in order for the system to be true.
- **Example:**

$$x + 2y = 5$$
$$3x + 4y = 11$$

This system has the unique solution of $x = 1$ and $y = 2$. We can see that this is the case when we plot each of these equations.
Notice in the *only* point in the graph where these two lines intersect is at $(1,2)$.

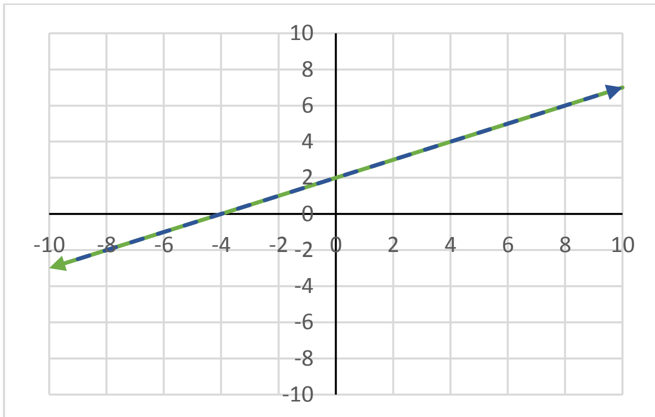
- **Dependent Systems:**
    - A system is said to be dependent if there are infinitely many solutions to the system. This means that no matter which x or y-value you choose, all of the equations in the system will be satisfied.

- **Example:**

$$3x - 6y = -12$$
$$-x + 2y = 4$$

This system has infinitely many solutions because any x and y values that satisfies one equation will also satisfy the other. We can see this more clearly if we graph both equations.

Notice the two equations form the same line (which has infinitely many points).

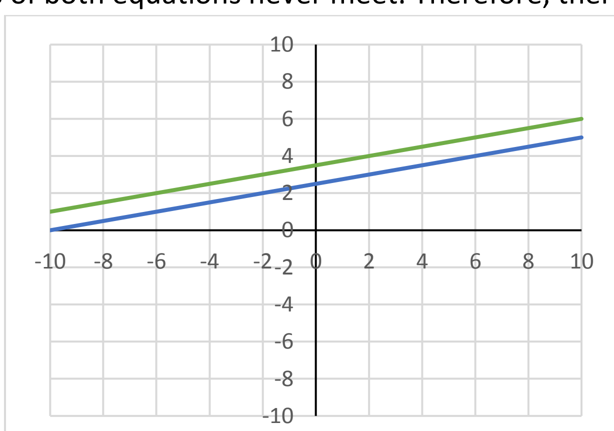


- **<u>Inconsistent Systems:</u>**
  - Inconsistent systems are system that have no solution to them. In other words, there is no pair of x and y-values that satisfy one equation that will also satisfy the other.
- **Example:**

$$\frac{1}{2}x - 2y = 5$$
$$x - 4y = 7$$

This system has no solution. We can see that this is the case when we plot each of these equations.

Notice that the lines of both equations never meet. Therefore, there is no solution.



Solving Systems Using Substitution

- Now that we know what systems are and the different types they could be, now we can focus on how we can obtain the solutions. The first method to accomplish this is called the substitution method.
- **How to Solve Systems Using Substitution:**
    - **1:** Using only *one* of the equations in the system, solve for one of the two unknown variables.
    - **2:** Using the equation that you did not use in step 1, replace the variable you solved for with the new expression from the previous step.
    - **3:** Solve for the unknown variable.
    - **4:** Go back to one of the original equations and solve for the other using the solution you just found.
- **Example:**

$$x - 2y = 4$$
$$3y + x = 19$$

Step 1: Solve the first equation for x.

$$x - 2y = 4$$
$$+2y \quad + 2y$$
$$x = 4 + 2y$$

Note: We could solve either equation for either variable. Solving for x in the first equation is just one of our options.

Step 2: Plug in $4 + 2y$ for x in the equation that we did not use.

$$3y + x = 19$$
$$3y + (4 + 2y) = 19$$
$$5y + 4 = 19$$

Step 3: Solve for y.

$$5y + 4 = 19$$
$$-4 \quad - 4$$
$$\frac{5y}{5} = \frac{15}{5}$$
$$y = 3$$

Step 4: Go back to one of the original equations and solve for x using the y-value we just found.
For this example, we will use the first equation to solve for x.

$$x - 2y = 4$$
$$x - 2(3) = 4$$
$$x - 6 = 4$$
$$+6 \quad + 6$$
$$x = 10$$

Therefore, the solution to this system is $x = 10$ and $y = 3$.

Solving Systems Using Elimination

- Solving by substitution is just one way that we can solve a system of equations. The other method is called the elimination method.
- **<u>How to Solve Systems Using Elimination:</u>**
  - **1:** Pick one of the variables that you would like to eliminate.
  - **2:** Multiply through one (or both) of the equations until either
    - The coefficients of the variable you are eliminating are the same or
    - The coefficients of the variable you are eliminating are the same, but with opposite signs (one is positive and one is negative).
  - **3:** Add the equations (if the signs are opposite) or subtract the equations (if the signs are the same).
  - **4:** Solve for the remaining variable.
  - **5:** Go back to one of the original equations and solve for the other using the solution you just found.
- **Example:**

$$2x + 3y = 14$$
$$3x - 5y = 2$$

Step 1: Even though we could pick any variable to eliminate, let's eliminate x in this example.

Step 2: In order to eliminate x, we are going to multiply through the first equation by 3 and the second equation by 2 in order to get the coefficients to be the same.

$$(2x + 3y) \cdot 3 = (14) \cdot 3$$
$$(3x - 5y) \cdot 2 = (2) \cdot 2$$

$$6x + 9y = 42$$
$$6x - 10y = 4$$

Step 3: Subtract the equations.

$$\begin{array}{r} 6x + 9y = 42 \\ \underline{-(6x - 10y = 4)} \\ 0 + 19y = 38 \end{array}$$

Step 4: Solve for y.

$$\frac{19y}{19} = \frac{38}{19}$$
$$y = 2$$

Step 5: Go back to one of the original equations and solve for x using the y-value we just found.
For this example, we will use the second equation to solve for x.

$$3x - 5y = 2$$
$$3x - 5(2) = 2$$
$$3x - 10 = 2$$
$$+10 \qquad +10$$
$$3x = 12$$
$$\frac{3x}{3} = \frac{12}{3}$$
$$x = 4$$

Therefore, the solution to this system is $x = 4$ and $y = 2$.

Systems of Linear Equations in Three Variables

How to Solve a System of Linear Equations in Three Variables

- **Steps:**
  - **1.** Using two of the three given equations, eliminate one of the variables.
  - **2.** Using a different set of two equations from the given three, eliminate the *same* variable that you eliminated in step one.
  - **3.** Use these two equations (which are now in two variables) and solve the system.
  - **4.** Use the values you find in step 3 to find the third variable using one of the original equations.
- **Example:** Solve the following system

$$x + 2y + z = 10$$
$$2x - 6y - z = -15$$
$$3x + 2y + 2z = 17$$

Step 1: Use the first and second equation to eliminate the variable z.

$$x + 2y + z = 10$$
$$\underline{+(2x - 6y - z = -15)}$$
$$3x - 4y = -5$$

Step 2: Use the second and third equation and eliminate the variable z.

$$(2x - 6y - z) \cdot 2 = (-15) \cdot 2$$
$$3x + 2y + 2z = 17$$

$$\begin{array}{r} 4x - 12y - 2z = -30 \\ \underline{+(3x + 2y + 2z = 17)} \\ 7x - 10y = -13 \end{array}$$

Step 3: Use the two equations that we just computed to solve for x and y.

$$3x - 4y = -5$$

$$7x - 10y = -13$$

We could use either elimination or substitution. For this example, let's use elimination to eliminate x.

$$(3x - 4y) \cdot 7 = (-5) \cdot 7$$

$$(7x - 10y) \cdot 3 = (-13) \cdot 3$$

$$\begin{array}{r} 21x - 28y = -35 \\ \underline{-(21x - 30y = -39)} \\ 2y = 4 \\ \overline{2} \quad \overline{2} \\ y = 2 \end{array}$$

We will find x by plugging $y = 2$ into $3x - 4y = -5$.

$$3x - 4y = -5$$

$$3x - 4(2) = -5$$

$$3x - 8 = -5$$

$$+8 \quad +8$$

$$3x = 3$$

$$\overline{3} \quad \overline{3}$$

$$x = 1$$

Step 4: Using $x = 1$ and $y = 2$, we will use the first equation in the original system of three equations to solve for z.

$$x + 2y + z = 10$$

$$1 + 2(2) + z = 10$$

$$5 + z = 10$$

$$-5 \quad -5$$

$$z = 5$$

Therefore the solution to this system is $x = 1$, $y = 2$, and $z = 5$.

Types of Solutions

- **Independent Systems:**
  - A system of equations in three variables is said to be independent if there is only one unique solution to solve the system. Note that the example above is an independent system.

- **Dependent Systems:**
  - A system of equations in three variables is said to be dependent if there are infinitely many solutions to the system. If this is the case, there will eventually be a step in solving the system that will result in $c = c$ where c is some constant.

- **Inconsistent Systems:**
  - A system of equations in three variables is said to be inconsistent if there is no solution to the system. If this is the case, there will eventually be a step in solving the system that will result in $k = c$ where k and c are constants with *different* values.