# Real Numbers

## Key Definitions

- **Integer:** An integer is a whole number (Example: …, $-3, -2, -1, 0, 1, 2, 3$,…)
- **Rational Number:** A rational number is a number that can be written as a fraction of two integers, $\frac{a}{b}$, where the integer $a$ is called the numerator and the integer $b$ is called the denominator and where $b$ is not 0. In decimal form, a rational number does not have repeating decimal places. $\left(\text{Example:} \frac{3}{2} = 1.5\,;\ \frac{10}{5} = 2\right)$
- **Irrational Number:** An irrational number is a number that has repeating decimal places when in decimal form. $\left(\text{Example: } \frac{2}{3} = .66666\ldots; \sqrt{3}\right)$
- **Variable:** A variable is a letter that represents a specific number that is often unknown. (Example: In the expression $x + 6, x$ is a $variable$)
- **Constant:** A constant is a number that is both fixed and known. (Example: In the expression $x + 6, 6$ is a $constant$)
- **Coefficient:** A coefficient is a constant that is multiplied by a variable. (Example: In the expression $5x + 3, 5$ is a $coefficient$)

## Decimal Approximation

- **Rounding:** When a number is in decimal form, it can be approximated to a given decimal place. When the next digit to the right is 5 or greater, round up, otherwise, round down.
  **Example:** Round 9.7836 to three decimal places:
  To round, look to the right of 3. Because "6" is greater than 5, round up by adding 1 to the 3: 9.784

## Order of Operations

- **Order of Operations:** The four arithmetic operations are addition, subtraction, multiplication, and division. When solving an expression involving more than one operation the correct order of operations is as follows: 1) start with the innermost parentheses (or brackets) and work outward; 2) multiply and divide, working left to right, and 3) add and subtract, working left to right.
  **Example:** Simplify the following expression:

$$3\left[2\cdot\left(4-\frac{6}{3}\right)\right]-5+3$$

Step 1: Solve inside the parentheses:

$\mathbf{Divide\ \frac{6}{3}} \rightarrow 3[2\cdot\left(4-\frac{6}{3}\right)]-5+3 = 3[2\cdot(4-2)]-5+3$

$\mathbf{Subtract\ 4-2} \rightarrow 3[2\cdot(4-2)]-5+3 = 3[2\cdot(2)]-5+3$

Step 2: Solve inside the brackets: $3[2\cdot 2]-5+3$

$\mathbf{Multiply\ 2\cdot 2} \rightarrow 3[2\cdot 2]-5+3 = 3[4]-5+3 = 3\cdot 4-5+3$

Step 3: Solve $3\cdot 4-5+3$

$\mathbf{Multiply\ 3\cdot 4} \rightarrow 3\cdot 4-5+3 = 12-5+3$

**Subtract** $\mathbf{12-5} \rightarrow 12-5+3=7+3$
**Add** $\mathbf{7+3} \rightarrow 7+3=10$

# Algebraic Expressions

- **Algebraic Expression:** A combination of variables and constants joined together by basic operations like addition, subtraction, multiplication, and division. (Example: $3x-7$)
- **Term:** A quantity within an algebraic expression separated from other quantities by addition or subtraction.
  **Example:** The above example has two terms $3x$ and 7
- **Evaluating an Algebraic Expression:** Replacing a variable with its value when that value is given.
  **Example:** Evaluate the following algebraic expression for $y=1$

$$\frac{(6y-2(y-1))}{(-1-y(3-y))}$$

Step 1: Replace each $y$ with 1:

$$\frac{6(1)-2(1-1)}{-1-(1)(3-1)}$$

Step 2: Follow order of operations:

**Solve inside inner parentheses** $\rightarrow \frac{(6(1)-2(1-1))}{(-1-(1)(3-1))}=\frac{(6(1)-2(0))}{(-1-(1)(2))}$

**Multiply** $\rightarrow \frac{(6(1)-2(0))}{(-1-(1)(2))}=\frac{(6-0)}{(-1-2)}$

**Subtract inside parentheses** $\rightarrow \frac{(6-0)}{(-1-2)}=\frac{6}{-3}$

**Divide** $\mathbf{\frac{6}{-3}} \rightarrow \frac{6}{-3}=-2$

# Properties of Real Numbers

- **Commutative property of addition:** Two or more real numbers can be added in any order.

$$6x+2+3=2+6x+3=2+3+6x$$

- **Commutative property of multiplication:** Two or more real numbers can be multiplied in any order.

$$y \cdot x \cdot 6=6xy$$

- **Associative property of addition:** When three or more real numbers are added, it does not matter what order the numbers are added.

$$(x+3)+8=x+(3+8)$$

- **Associative property of multiplication:** When three or more real numbers are multiplied it does not matter what order the numbers are multiplied.

$$(5y)x=5(yx)$$

- **Distributive property:** Multiplication is distributed over *all* terms of the sums or differences within the parentheses.

$$2(x+6)=2x+12$$
$$2(x-6)=2x-12$$

- **Additive identity property:** Adding zero to any number gives back the same real number.
  $x + 0 = x$
- **Multiplicative identity property:** Multiplying any number by 1 gives the same real number.
  $(4x) \cdot 1 = 4x$
- **Additive inverse property:** Adding a real number and its additive inverse (or opposite) gives zero.
  $2x + (-2x) = 0$
- **Multiplicative inverse property:** Multiplying a real number (not zero) and its multiplicative inverse (or reciprocal) gives one.
  $x \cdot \frac{1}{x} = 1$

- **Properties of Negative Numbers:**
  - A negative number multiplied by a positive number is a negative number
    $$(\text{ex: } (-2)(3) = -6)$$
  - A negative number divided by a positive number is a negative number
    $$\left(\text{ex: } \frac{(-8)}{(4)} = -2\right)$$
    OR a positive number divided by a negative number is a negative number
    $$\left(\text{ex: } \frac{(8)}{(-4)} = -2\right)$$
  - A negative number multiplied by a negative number is a positive number
    $$(\text{ex: } (-5)(-3) = 15)$$
  - A negative number divided by a negative number is a positive number
    $$\left(\text{ex: } \frac{-20}{-5} = 4\right)$$
  - Subtracting a negative number is the same as adding a positive number
    $$(\text{ex: } 3 - (-2) = 3 + 2 = 5)$$
  - A negative sign in front of an expression must be distributed throughout the expression
    $$\left(\begin{array}{c}\text{ex: } -(2+5) = -2 + (-5) = -7 \\ \text{or } -(3-6) = -3 - (-6) = -3 + 6 = -3\end{array}\right)$$
- **Properties of Fractions:**
  - Multiplying fractions:
    $$\left(\frac{2}{3}\right) \cdot \left(\frac{4}{5}\right) = \frac{(2 \cdot 4)}{(3 \cdot 5)} = \frac{8}{15}$$
  - Adding fractions with the same denominator:
    $$\left(\frac{2}{3}\right) + \left(\frac{5}{3}\right) = \frac{(2+5)}{3} = \frac{7}{3} \quad or \quad \left(\frac{x}{4}\right) + \left(\frac{3}{4}\right) = \frac{(x+3)}{4}$$
  - Subtracting fractions with the same denominator:
    $$\left(\frac{7}{4}\right) - \left(\frac{5}{4}\right) = \frac{(7-5)}{4} = \frac{2}{4} \quad or \quad \left(\frac{x}{3}\right) - \left(\frac{2}{3}\right) = \frac{(x-2)}{3}$$
  - Adding fractions with different denominators:
    $$\left(\frac{3}{8}\right) + \left(\frac{1}{2}\right) = \left(\frac{(3 \cdot 2)}{(8 \cdot 2)}\right) + \left(\frac{(1 \cdot 8)}{(2 \cdot 8)}\right) = \frac{(3 \cdot 2) + (1 \cdot 8)}{(8 \cdot 2)} = \frac{(6+8)}{16} = \frac{14}{16}$$

- Subtracting fractions with different denominators:

$$\left(\frac{3}{4}\right)-\left(\frac{1}{8}\right)=\frac{(3\cdot 8)-\ (1\cdot 4)}{(4\cdot 8)}=\frac{(24-4)}{32}=\frac{20}{32}$$

- Dividing by a fraction is the same as multiplying by its reciprocal:

$$\frac{\left(\frac{2}{3}\right)}{\left(\frac{4}{5}\right)}=\left(\frac{2}{3}\right)\cdot\left(\frac{5}{4}\right)=\frac{(2\cdot 5)}{(3\cdot 4)}=\frac{10}{12}$$

- **Properties of Zero:**
  - A real number multiplied by zero is zero: $0\cdot x=0$
  - Zero divided by a nonzero real number is zero: $\frac{0}{x}=0$
  - A real number divided by zero is undefined: $\frac{x}{0}$ is undefined
- **Zero Product Property:** If the product of two numbers is zero, then one or both numbers are zero:

$$\text{If } x(x+5)=0, \text{then } x=0 \text{ or } x+5=0 \text{ (if } x+5=0 \text{ then } x=-5)$$

# Complex Numbers

## Key Definitions

- **Imaginary Unit, *i*:** The imaginary unit is denoted by the letter *i* and is defined as:

$$i=\sqrt{-1} \quad \text{where} \quad i^2=-1$$

- Recall that for positive real numbers *a* and *b* we defined the square root as:

$$b=\sqrt{a} \quad \text{which means} \quad b^2=a$$

  Similarly we define the square root of a negative number as:

$$\sqrt{-a}=i\sqrt{a} \quad \text{since} \quad (i\sqrt{a})^2=i^2a=-1\cdot a=-a$$

- **Complex Number:** A complex number is number that involves both real and imaginary numbers written in the form $a+bi$ where *a* and *b* are real numbers and *i* is the imaginary unit. We say that *a* is the real part of the complex number and *bi* is the imaginary part of the complex number.

## Operations on Complex Numbers

- Complex numbers are treated in a similar way to binomials. We can add, subtract, and multiply complex numbers the same way we performed these operations on binomials.
- **Adding and Subtracting:** Combine similar terms (i.e. real parts with real parts and imaginary parts with imaginary parts).
  **Example:** $(3-2i)+(-1+i)=3-2i-1+i=(3-1)+(-2i+i)=2-i$
  **Example:** $(2-i)-(3-4i)=2-i-3+4i=(2-3)+(-i+4i)=-1+3i$
- **Multiplying:** Follow the same procedure as binomials (FOIL)
  **Example:**

$$(3-i)(2+i)=(3)(2)+(3)(i)+(-i)(2)+(-i)(i)=6+3i-2i-i^2$$
$$=6+3i-2i-(-1)=(6+1)+(3i-2i) \quad =7+i$$

## Conjugates

- **<u>Complex Conjugate:</u>** If the standard form of a complex number is $a + bi$, then the conjugate of that complex number is $a - bi$.
- The product of a complex number and its conjugate results in a real number because the imaginary terms cancel out.
  Example: If $z = a + bi$ and $\bar{z} = a - bi$, then
  $$z\bar{z} = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2$$
- **<u>Dividing:</u>** In order to divide two complex numbers, you must first multiply the numerator and denominator by the conjugate of the denominator.
  **Example:**
  $$\frac{3-4i}{1+2i} = \frac{3-4i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{(3-4i)(1-2i)}{(1+2i)(1-2i)} = \frac{(3)(1)+(3)(-2i)+(-4i)(1)+(-4i)(-2i)}{(1)^2+(2i)^2}$$
  $$= \frac{3-6i-4i+8i^2}{1+4i^2} = \frac{3-6i-4i+8(-1)}{1+4(-1)} = \frac{(3-8)+(-6i-4i)}{1-4} = \frac{-5-10i}{-3} = \frac{5+10i}{3}$$

## Imaginary Units and Exponents

- **<u>Raising Complex Numbers to Exponents:</u>** Note that *i* raised to the fourth power is 1. In simplifying imaginary numbers, we factor out *i* raised to the largest multiple of 4: $i = \sqrt{-1}$
  $i^2 = (\sqrt{-1})^2 = -1$

  $i^3 = i^2 \cdot i = -1 \cdot i = -i$

  $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

  $i^5 = i^4 \cdot i = 1 \cdot i = i$

  $i^6 = i^4 \cdot i^2 = 1 \cdot -1 = -1$

  $i^7 = i^4 \cdot i^3 = 1 \cdot -i = -i$

  $i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$

  **Example:** $i^{30} = i^{28} \cdot i^2 = (i^4)^7 \cdot i^2 = (1)^7 \cdot -1 = 1 \cdot -1 = -1$
  **Example:** Find $(2 + 3i)^3$
  **[**Recall the formula for cubing a binomial: $(a^3 + b^3) = a^3 + 3a^2b + 3ab^2 + b^3$**]**
  $$(2+3i)^3 = 2^3 + (3)(2^2)(3i) + (3)(2)(3i)^2 + (3i)^3 = 8 + 36i + 54i^2 + 27i^3$$
  $$= 8 + 36i + 54(-1) + 27(-i) = (8 - 54) + (36i - 27i) = -46 - 9i$$