Polynomials

Key Definitions

- **Degree:** Degree refers to the value of a terms exponent. The degree of a polynomial is the value of the highest exponent of any single term.
  Example: The polynomial $5x^5 + 3x^4 + x + 7$ is a $fifth$ degree polynomial
  The polynomial $6x - 2$ is a $first$ degree polynomial
- **Polynomial:** A polynomial is an algebraic expression of the form:
  $$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$
  where $a_0, a_1, a_2, \ldots, a_n$ are real numbers, with $a_n \neq 0$, and n is a nonnegative integer. The degree of the polynomial is n.
  Example: $3x^4 + 6x^3 + 9x^2 - 2x + 5$ is a polynomial where n is 4
  $5x^7 + x^3 - 3x + 8$ is a polynomial where a_6, a_5, a_4, and $a_2 = 0$ and n is 7
- **Standard form:** The standard form of a polynomial orders its terms by decreasing degree.
  Example: $3x - 2x^3 + x^5 - 7$ in standard form is $x^5 - 2x^3 + 3x - 7$
- **Like terms:** Like terms are terms that have the same variable and the same degree
  Example: $4x^3$ and $4x^2$ are not $like\ terms$, but $2x^4$ and $3x^4$ are.

Operations with Polynomials

- **Adding:** Adding polynomials involves finding like terms and combining them by adding their coefficients.
  Example: Find the sum and simplify:
  $$(4x^3 + 7x^2 - 4) + (3x^2 - x + 5)$$
  Step 1: Eliminate parentheses: $4x^3 + 7x^2 - 4 + 3x^2 - x + 5$
  Step 2: Identify like terms: $4x^3 + 7x^2 - 4 + 3x^2 - x + 5$
  Step 3: Combine like terms: $4x^3 + 10x^2 + 1 - x$
  Step 4: Write in standard form: $4x^3 + 10x^2 - x + 1$
- **Subtracting:** Subtracting polynomials involves distributing the negative to each term of the second polynomial, then finding and combining like terms.
  Example: Find the difference and simplify:
  $$(6x^5 - 3 + 4x^2) - (3x^5 - 5x + 4)$$
  Step 1: Eliminate parentheses:
  Eliminate 1st pair → $6x^5 - 3 + 4x^2 - (3x^5 - 5x + 4)$
  Distribute negative → $6x^5 - 3 + 4x^2 - 3x^5 + 5x - 4$
  Step 2: Identify like terms: $6x^5 - 3 + 4x^2 - 3x^5 + 5x - 4$
  Step 3: Combine like terms: $3x^5 - 7 + 4x^2 + 5x$
  Step 4: Write in standard form: $3x^5 + 4x^2 + 5x - 7$
- **Multiplying:** Multiplying polynomials involves using the distributive property repeatedly. Each term in the first polynomial is multiplied by each term in the second polynomial. After

multiplication, the polynomial is simplified by combining similar terms (i.e. terms with the same power).

Example: Find the product and simplify:

$$(2x^2 - x)(6x^3 + 3x + 4)$$

Step 1: Use the distributive property:

$$(2x^2)(6x^3) + (2x^2)(3x) + (2x^2)(4) + (-x)(6x^3) + (-x)(3x) + (-x)(4)$$

Step 2: Multiply:

$$12x^5 + 6x^3 + 8x^2 - 6x^4 - 3x^2 - 4x$$

Step 3: Combine similar terms:

$$12x^5 - 6x^4 + 6x^3 + 8x^2 - 3x^2 - 4x = 12x^5 - 6x^4 + 6x^3 + 5x^2 - 4x$$

Answer: $12x^5 - 6x^4 + 6x^3 + 5x^2 - 4x$

FOILing

- The **FOIL** method applies the distributive property in an easy to remember acronym by finding the products of the **F**irst terms, **O**uter terms, **I**nner terms, and **L**ast terms when multiplying two binomials.

  Example: Find the product and simplify: $(2x - 1)(4x + 3)$

  Step 1: Multiply the **F**irst terms: $(2x)(4x) = 8x^2$

  Step 2: Multiply the **O**uter terms: $(2x)(3) = 6x$

  Step 3: Multiply the **I**nner terms: $(-1)(4x) = -4x$

  Step 4: Multiply the **L**ast terms: $(-1)(3) = -3$

  Step 5: Add the products from steps 1-4: $8x^2 - 4x + 6x - 3$

  Step 6: Combine like terms: $8x^2 + 2x - 3$

Special Products

- **Difference of Two Squares:** $(a + b)(a - b) = a^2 - b^2$

  Example: Find the product and simplify: $(x + 2)(x - 2) = (x)^2 - (2)^2 = x^2 - 4$

- **Perfect Squares:**
  - Square of a Binomial Sum: $(a + b)^2 = a^2 + 2ab + b^2$

    Example: Find the following: $(2x + 3)^2 = (2x)^2 + 2(2x)(3) + (3)^2 = 4x^2 + 12x + 9$
  - Square of a Binomial Difference: $(a - b)^2 = a^2 - 2ab + b^2$

    Example: $(4x - 1)^2 = (4x)^2 - 2(4x)(1) + (1)^2 = 16x^2 - 8x + 1$

- **Perfect Cubes:**
  - Cube of a Binomial Sum: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

    Example: $(3x + 2)^3 = (3x)^3 + 3(3x)^2(2) + 3(3x)(2)^2 + (2)^3$

    $$= (3^3)(x)^3 + (3 \cdot 3^2 \cdot 2)(x)^2 + (3 \cdot 3 \cdot 2^2)(x) + 2^3$$

    $$= 27x^3 + 54x^2 + 36x + 8$$
  - Cube of a Binomial Difference: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

    Example: $(2x - 4)^3 = (2x)^3 - 3(2x)^2(4) + 3(2x)(4)^2 - (4)^3$

    $$= (2^3)(x)^3 - (3 \cdot 2^2 \cdot 4)(x)^2 + (3 \cdot 2 \cdot 4^2)(x) - 4^3$$

    $$= 27x^3 + 54x^2 + 36x + 8$$

Factoring Polynomials

Key Definitions

- **Factor:** When a number or expression is written as a product, the quantities that are multiplied together are *factors*.
  Example: Because $6 = 3 \cdot 2$ we call 3 and 2 *factors* of 6; because $x^2 + 3x + 2 = (x+2)(x+1)$ we call $(x+2)$ and $(x+1)$ *factors* of $x^2 + 3x + 2$
- **Greatest Common Factor:** The greatest common factor of a polynomial in x is of the form ax^k where a is the largest factor common to all the polynomial coefficients and k is the smallest exponent of x found in all of the terms of the polynomial.
  Example: The greatest common factor of $4x^2 + 16x$ is $4x$

Factoring

- **Difference of Two Squares:** A polynomial of the following form can be factored as follows: $a^2 - b^2 = (a+b)(a-b)$
  Example: $x^2 - 9 = x^2 - 3^2 = (x+3)(x-3)$
- **Perfect Squares:** A polynomial of the following form can be factored as follows:
  $a^2 + 2ab + b^2 = (a+b)(a+b) = (a+b)^2$ or $a^2 - 2ab + b^2 = (a-b)(a-b) = (a-b)^2$
  Example: $x^2 + 6x + 9 = x^2 + 2(3)(x) + 3^2 = (x+3)(x+3) = (x+3)^2$
  $9x^2 - 12x + 4 = (3x)^2 - 2(3x)(2) + 2^2 = (3x-2)(3x-2) = (3x-2)^2$
- **Sum of Two Cubes**: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  Example:
  $27x^3 + 8 = (3x)^3 + 2^3 = (3x+2)((3x)^2 - (3x)(2) + 2^2) = (3x+2)(9x^2 - 6x + 4)$
- **Difference of Two Cubes**: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
  Example: $8x^3 - 1 = (2x)^3 - 1^3 = (2x-1)((2x)^2 + (2x)(1) + 1^2) = (2x-1)(4x^2 + 2x + 1)$

Factoring by Grouping

- **Topic:** For polynomials with more than three terms we first look for a common factor to all terms. If there is no common factor to all terms of the polynomial, we look for a group of terms that have a common factor.
  Example:
  $x^3 - x^2 + 2x - 2 = (x^3 - x^2) + (2x - 2) = x^2(x-1) + 2(x-1) = (x-1)(x^2+2)$

Strategy for Factoring Polynomials

1. Factor out the greatest common factor (monomial).
2. Identify any special polynomial forms and apply factoring formulas (i.e. sum of cubes, difference of two squares, etc.)
3. Factor a trinomial into a product of two binomials: $(ax+b)(cx+d)$
4. Factor by grouping.

Rational Expressions

Key Definitions

- **Topic:** Recall that a rational number is the ratio of two integers with the denominator not equal to zero. We apply this same concept to polynomials so that the ratio of two polynomials is called a rational expression.

  Example: Rational Numbers: $\frac{3}{7}, \frac{4}{9}, \frac{7}{11}$ Rational Expressions: $\frac{7}{2x-1}, \frac{3x-1}{x-2}, \frac{2x^2}{x^2+1}$

Domain

- **Topic:** Since the denominator cannot be equal to zero in a rational expression, there are restrictions to what values of x we are allowed to use. This is called the domain of an algebraic expression.
- **Polynomials:** The domain for a polynomial is any real number as we can use any value for x because there is no denominator.
- **Rational Expressions:** The domain for a rational expression is dependent upon the denominator. Any value for x that would make the denominator zero must be excluded.

Examples		
Algebraic Expression	Domain	Note
$2x^2 - 3x + 1$	All real numbers	The domain of all polynomials is the set of all real numbers
$\frac{3x+1}{x-6}$	All real numbers except $x = 6$	When $x = 6$, the rational expression is undefined.
$\frac{4x}{x^2+3}$	All real numbers	There are no real numbers that will result in the denominator being equal to zero.
$\frac{5x+2}{x}$	All real numbers except $x = 0$	When $x = 0$, the rational expression is undefined.

Simplifying Rational Expressions

- **Topic:** Recall that a fraction is reduced when it is written with no common factors

$$\frac{16}{12} = \frac{4 \cdot 4}{4 \cdot 3} = \frac{4}{4} \cdot \frac{4}{3} = 1 \cdot \frac{4}{3} = \frac{4}{3}$$

Similarly, rational expressions are simplified by reducing to lowest terms if the numerator and denominator have no common factors.

- **<u>Steps to Simplifying a Rational Expression</u>**:
  - Factor the numerator and denominator completely.
  - State any domain restrictions.
  - Cancel (divide out) the common factors in the numerator and denominator.

Example: Simplify $\frac{x^2-4}{2x+4}$ and state any domain restrictions.

$$\frac{x^2-4}{2x+4} = \frac{(x-2)(x+2)}{2(x+2)} = \frac{x-2}{2} \cdot \frac{x+2}{x+2} = \frac{x-2}{2} \cdot 1 = \frac{x-2}{2}$$

The domain is $x \neq -2$.

Note: When finding the domain, look at the rational expression after factoring and before cancelling.

- **<u>Multiplying and Dividing Rational Expressions:</u>**
  - Factor all numerators and denominators completely.
  - State any domain restrictions.
  - Rewrite division as multiplication by a reciprocal.
  - State any additional domain restrictions.
  - Divide the numerators and denominators by any common factors.
  - Multiply the remaining numerators and denominators, respectively.
    (Note: For multiplication, exclude steps 3 and 4.)

Example: Divide and simplify $\frac{3x+1}{4x^2+4x} \div \frac{9x+3}{x^3+3x^2+2x}$

1. $\frac{3x+1}{4x^2+4x} \div \frac{9x+3}{x^3+3x^2+2x} = \frac{3x+1}{4x(x+1)} \div \frac{3(3x+1)}{x(x+1)(x+2)}$
2. $\frac{3x+1}{4x(x+1)} \div \frac{3(3x+1)}{x(x+1)(x+2)} = \frac{3x+1}{4x(x+1)} \cdot \frac{x(x+1)(x+2)}{3(3x+1)}$
3. $= \frac{\cancel{3x+1}}{4\cancel{x(x+1)}} \cdot \frac{\cancel{x(x+1)}(x+2)}{3\cancel{(3x+1)}}$
4. $= \frac{1}{4} \cdot \frac{x+2}{3} = \frac{1 \cdot (x+2)}{4 \cdot 3} = \frac{x+2}{12}$

Least Common Multiple

- **<u>Topic:</u>** Recall that fractions can only be added or subtracted when their denominators are the same.

$$\frac{4}{5} + \frac{2}{5} = \frac{4+2}{5} = \frac{6}{5}$$

When the denominators are not the same, the least common multiple (LCM) of the denominators is used to make them the same.

$$\frac{5}{6} - \frac{1}{4} = \frac{5}{2 \cdot 3} - \frac{1}{2 \cdot 2} = \left(\frac{2}{2} \cdot \frac{5}{2 \cdot 3}\right) - \left(\frac{1}{2 \cdot 2} \cdot \frac{3}{3}\right) = \frac{10}{2 \cdot 2 \cdot 3} - \frac{3}{2 \cdot 2 \cdot 3} = \frac{7}{12}$$

The LCM is the smallest number that is divisible by two numbers. As the example above shows, the LCM combines the factors of both numbers.

Like fractions, rational expressions are added or subtracted only when their denominators are the same.

- **<u>Steps to Adding or Subtracting Rational Expressions</u>:**
  - Factor the denominators of the rational expressions completely.
  - Write each denominator as the product of distinct factors raised to powers (i.e. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$ would be written as $2^3 \cdot 3 \cdot 5^2$)
  - Look at the denominators and find the highest power of each distinct factor, then find the product of the distinct factors raised to their highest power. This product is the least common multiple (LCM) of the denominators.
  - Write each rational expression using the LCM for each denominator.
  - Add or subtract the resulting numerators.
  - Factor the resulting numerator to check for common factors.

Example: Perform the indicated operation and write in simplified form:

$$\frac{2x}{3x-9} - \frac{4x-7}{x^2-6x-9} = \frac{2x}{3(x-3)} - \frac{4x-7}{(x-3)(x-3)} = \frac{2x(x-3)}{3(x-3)(x-3)} - \frac{4x-7}{(x-3)^2}$$

$$= \frac{2x(x-3)}{3(x-3)^2} - \frac{3(4x-7)}{3(x-3)^2} = \frac{2x^2-6x-12x+21}{3(x-3)^2}$$

$$= \frac{2x^2-18x+21}{3(x-3)^2} \text{ where } x \neq 3$$

$$LCM = 3(x-3)^2$$