Lines

Key Definitions

- **General Form of a Line**: $Ax + By = C$

  A, B, and C are constants, and x and y are variables (ex: $2x - 3y = 5$).
- **Slope of a Line**: $m = \frac{y_2 - y_1}{x_2 - x_1}$ using points (x_1, y_1) and (x_2, y_2), in applications as the rate of change
- **Slope-Intercept Form of a Line**: $y = mx + b$ where m is the slope and b is the y-intercept of the line (must be a non-vertical line)
- **Point-Slope Form of a Line**: $y - y_1 = m(x - x_1)$ where m is the slope and (x_1, y_1) is a point on the line
- **Horizontal Lines**: have a slope of zero and have the form $y = b$
- **Vertical Lines:** have an undefined slope and have the form $x = a$
- **Parallel Lines:** two lines are parallel if and only if their slopes are equal $(m_1 = m_2)$
- **Perpendicular Lines:** two lines are perpendicular if and only if their slopes are negative reciprocals of each other $(m_1 = \frac{-1}{m_2})$

Finding the Slope of a Line

- **To find the slope of a line, find two points on the line and use $m = \frac{y_2 - y_1}{x_2 - x_1}$**
- **Examples: For each of the following pair of points, find the line's slope.**

  a. (-3,4) and (4,6)

  $$m = \frac{6-4}{4-(-3)} = \frac{2}{7}$$

  b. (2,5) and (3,4)

  $$m = \frac{4-5}{3-2} = \frac{-1}{1} = -1$$

  c. (6,5) and (1,5)

  $$m = \frac{5-5}{1-6} = \frac{0}{-5} = 0$$

  Since the slope is zero, the line is horizontal. The equation of the line is $y = 5$ since both points have a 5 for their y-coordinate.

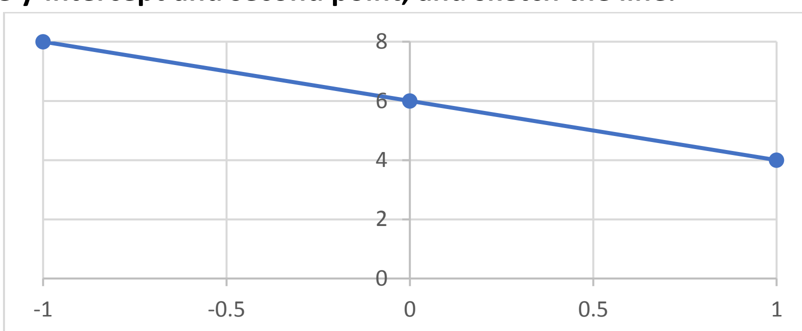
  d. (2,2) and (2,3)

  $$m = \frac{3-2}{2-2} = \frac{1}{0}, \text{which is undefined}$$

  Since the slope is undefined, the line is vertical. The equation of the line is $x = 2$ since both points have a 2 for their x-coordinate.

Slope-Intercept Form of a Line

- **To use the slope-intercept form of a line, find the slope and the y-intercept and use $y = mx + b$ where m is the slope and b is the y-intercept of the line.**
- **Example 1: Change $2x + y = 6$ to slope-intercept form and graph it.**
  Step 1: Isolate y:
  $$2x + y = 6$$
  Subtract 2x from both sides:
  $$y = 6 - 2x$$
  Rewrite the equation:
  $$y = -2x + 6$$
  The slope of the line is -2, and the y-intercept is (0,6).
  Step 2: Find a second point using the equation of the line:
  Use $x = 1$:
  $$y = -2(1) + 6 = -2 + 6 = 4$$
  The second point is (1,4).
  Step 3: Plot the y-intercept and second point, and sketch the line.



- **Example 2: Find the slope-intercept equation of the line with a slope of ¾ and a y-intercept of (0,4).**
  Step 1: Use the slope-intercept form:
  $$y = mx + b$$
  Step 2: Label the slope:
  $$m = \frac{3}{4}$$
  Step 3: Label the y-intercept:
  $$b = 4$$
  Step 4: Write the equation:
  $$y = \frac{3}{4}x + 4$$

Point-Slope Form of a Line

- **To use the point-slope form of a line, find the slope and a point on the line and use $\boldsymbol{y - y_1 = m(x - x_1)}$ where *m* is the slope and $\boldsymbol{(x_1, y_1)}$ is a point on the line.**
- **Example 1:** Find the equation of a line with a slope of 2 that passes through the point (3,7).
  Step 1: Use the point-slope form and enter the slope of 2 for m and the point (3,7) for $\boldsymbol{(x_1, y_1)}$.

$$\boldsymbol{y - y_1 = m(x - x_1)}$$
$$\boldsymbol{y - 7 = 2(x - 3)}$$

  Distribute the 2.

$$\boldsymbol{y - 7 = 2x - 6}$$

  Add 7 to both sides.

$$\boldsymbol{y = 2x + 1}$$

- **Example 2:** Find the equation of a line that passes through the points (2,4) and (-1,2).
  Step 1: Find the slope using the two points using the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{2 - 4}{-1 - 2} = \frac{-2}{-3} = \frac{2}{3}$$

  Step 2: Use the point-slope form and enter the slope of 2/3 for m and one of the points (-1,2).

$$\boldsymbol{y - y_1 = m(x - x_1)}$$
$$\boldsymbol{y - 2 = \frac{2}{3}(x - (-1))}$$

  Distribute the 2/3.

$$\boldsymbol{y - 2 = \frac{2}{3}(x + 1) = \frac{2}{3}x + \frac{2}{3}}$$

  Add 2 to both sides.

$$\boldsymbol{y = \frac{2}{3}x + 2\frac{2}{3}}$$

Parallel and Perpendicular Lines

- **To determine if two lines are parallel or perpendicular, change both lines to the slope-intercept form and compare the slopes based on the respective definitions.**
  - **Example 1**: Find the equation of the lines that goes through the point (2,7) and is parallel to $y = -2x + 6$
    Step 1: Find the slope of the line.
    Since the two lines are parallel, the line will have the same slope which is -2.
    Step 2: Use the point-slope form and enter the slope of -2 for m and the point (2,7).

$$\boldsymbol{y - y_1 = m(x - x_1)}$$
$$\boldsymbol{y - 7 = -2(x - 2)}$$

Distribute the -2.

$$y - 7 = -2x + 4$$

Add 7 to both sides.

$$y = -2x + 11$$

- **Example 2:** Find the equation of the line that goes through the point (4,1) and is perpendicular to $y = \frac{2}{3}x + 2\frac{2}{3}$.

  Step 1: Find the slope of the line.

Since the lines are perpendicular the slope of the line will be the negative reciprocal of the slope of the perpendicular line. The slope of the perpendicular line is 2/3. The negative reciprocal of 2/3 is -3/2.

Step 2: Use the point-slope form and enter the slope of -3/2 for m and the point (4,1).

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{-3}{2}(x - 4) = \frac{-3}{2}x + 6$$

Add 1 to both sides.

$$y = \frac{-3}{2}x + 7$$

Applications

- **Slope:** can be used as the rate of change
- **Example 1:** In 2010, a club had 60 members. In 2016, the club had 180 members. Find the slope of the line passing through these points. Describe what the slope represents.

  Step 1: Determine two points.

  The two points are (2010, 60) and (2016, 180).

  Step 2: Use the slope formula to find the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{180 - 60}{2016 - 2010} = \frac{120}{6} = 20$$

  Step 3: Describe the slope.

  Keep in mind that the numerator of the fraction has the number of members and the denominator of the fraction has the number of years. The slope is 20 and is the rate of change of the average number of members joining the club each year. For every 1 year, 20 members join the club.

- **Example 2:** Two friends hired the same cleaning service to clean their houses. One friend's house took 4 hours to clean and cost $100. The other friend's house took 7 hours to clean and cost $151. Assuming that a linear equation determines the cost of the cleaning, if your house takes 6 hours to clean, how much should this cleaning service charge you?

Step 1: Identify the question.
Find the linear equation for the cost of the cleaning service and determine how much they would charge for 6 hours.
Step 2: Make notes.
4 hours costs $100. 7 hours cost $151.
Step 3: Set up the equation.
Use the slope-intercept form of a line: $y = mx + b$
Use the points: (4, 100) and (7, 151)
Step 4: Solve the equation
Determine the slope based on the two points.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{151 - 100}{7 - 4} = \frac{51}{3} = 17$$

Substitute the slope into the equation.

$$y = 17x + b$$

Substitute one of the points into the equation to solve for the y-intercept.

Pick a point: (4, 100).

$$100 = 17(4) + b$$

$$100 = 68 + b$$

Subtract 68 from both sides.

$$32 = b$$

The linear equation is

$$y = 17x + 32$$

Substitute $x = 6$ into this equation to find the cost of a cleaning that takes 6 hours.

$$y = 17(6) + 32 = 102 + 32$$

$$y = 134$$

The 6-hour cleaning job should cost about $134.

Slope Visuals

If the line is	The slope is
Rising	Positive
Falling	Negative
Horizontal	Zero
Vertical	Undefined

Circles

Key Definitions

- **Circle:** the set of all points the same distance from a point called the center written as (h, k)
- **Radius:** the fixed distance from the points on the circle to the center, written as r
- **Standard Form of the Equation of a Circle:** $(x-h)^2+(y-k)^2=r^2$ where r is the radius of the circle and (h, k) is the center
- **Unit Circle:** a circle with radius 1 and center (0, 0), $x^2+y^2=1$
- **General Form of the Equation of a Circle**: $x^2+y^2+ax+by+c=0$

Equation of a Circle

- **Example 1: Identify the center and radius of this circle and graph it.**

$$(x+3)^2+(y-5)^2=9$$

Step 1: Rewrite the equation in standard form $(x-h)^2+(y-k)^2=r^2$

$$(x-(-3))^2+(y-5)^2=3^2$$

Step 2: Find h, k, and r using this equation: $h=-3, k=5, r=3$

The center (h, k) is (-3, 5), and the radius is 3. To graph the circle, first plot the center.

Then, plot four points 3 units away from the center

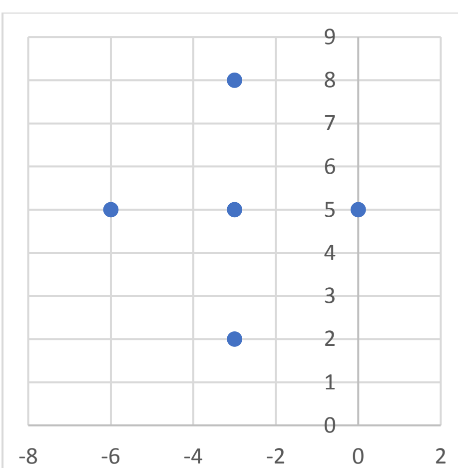
$$(-3,5+3)=(-3,8)$$

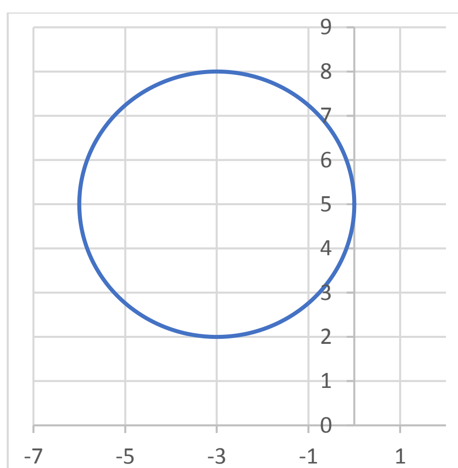
$$(-3,5-3)=(-3,2)$$

$$(-3+3,5)=(0,5)$$

$$(-3-3,5)=(-6,5)$$

Then, use those points to sketch the circle.





- **Example 2**: Write the equation of the circle with radius 4 and center (4, -9).
  Step 1: Use the equation of a circle with $r = 4 \; and \; (h, k) = (4, -9)$.
  $$(x-h)^2 + (y-k)^2 = r^2$$
  $$(x-4)^2 + \left(y-(-9)\right)^2 = 4^2$$
  $$(x-4)^2 + (y+9)^2 = 16$$
- **Example 3**: Write the equation of the circle with center (5, -6) and a point on the circle (9, 3) in general form.
  Step 1: Use the equation of the circle with the center $(h, k) = (5, -6)$.
  $$(x-h)^2 + (y-k)^2 = r^2$$
  $$(x-5)^2 + \left(y-(-9)\right)^2 = r^2$$
  $$(x-5)^2 + (y+9)^2 = r^2$$
  Step 2: Plug in the point (9, 3).
  $$(9-5)^2 + (3+9)^2 = r^2$$
  $$4^2 + 12^2 = r^2$$
  $$16 + 144 = r^2$$
  $$160 = r^2$$
  $$\sqrt{160} = r$$
  $$4\sqrt{10} = r$$
  Step 3: Use the radius r and center (h, k) to write the equation.
  $$(x-5)^2 + (y+9)^2 = r^2$$
  $$(x-5)^2 + (y+9)^2 = 4\sqrt{10}^2 = 160$$
  $$(x-5)^2 + (y+9)^2 = 160$$

  Step 4: Eliminate the parenthesis and simplify.
  $$(x-5)^2 + (y+9)^2 = 160$$
  $$x^2 - 10x + 25 + y^2 + 18y + 81 = 160$$
  $$x^2 + y^2 - 10x + 18y - 54 = 0$$
  This equation is in general form.

- **Example 4**: Find the center and radius of the circle with this equation:
  $$x^2 + y^2 + 8x + 2y - 28 = 0$$
  Step 1: To find the center and radius, we should change the equation to standard form.
  $$(x-h)^2 + (y-k)^2 = r^2$$

  Step 2: Group the x and y terms, respectively, on the left of the equal sign, and move the constants to the right of the equal sign.
  $$(x^2 + 8x) + (y^2 + 2y) = 28$$
  $$(x^2 + 8x + \_) + (y^2 + 2y + \_) = 28$$

  Step 3: Complete the squares for both the x and y expressions.

$$\frac{8^2}{2} = 4^2 = 16$$

$$\frac{2^2}{2} = 1^2 = 1$$

Step 4: Add these numbers to both sides.

$$(x^2 + 8x + 16) + (y^2 + 2y + 1) = 28 + 16 + 1$$
$$(x^2 + 8x + 16) + (y^2 + 2y + 1) = 45$$
$$(x + 4)^2 + (y + 1)^2 = 45$$

Step 5: Write in standard form.

$$(x - (-4))^2 + (y - (-1))^2 = (\sqrt{45})^2 = (3\sqrt{5})^2$$

The center is (-4, -1), and the radius is $3\sqrt{5}$.