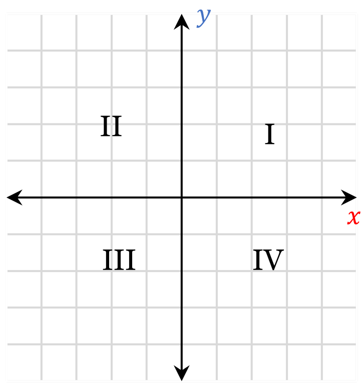
Cartesian Plane, Distance, and Midpoint

Cartesian Plane

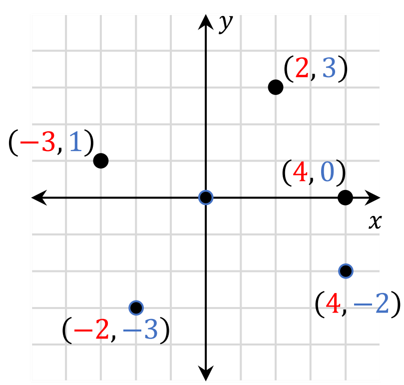
- **<u>Definition</u>:** The Cartesian plane is used to graph relationships between two quantities. It is formed by two perpendicular lines that intersect at a point called the *origin*. Those



two perpendicular lines are called axes. The horizontal axis is generally called the x-axis (x), and the vertical axis is generally called the y-axis (y). The axes divide the plane into four sections called quadrants (I, II, III, and IV).

- **<u>Ordered Pairs</u>:** Ordered pairs represent points in the Cartesian plane.

Example: $(2, 3)$, $(4,0)$, $(4, -2)$, $(-2, -3)$, and $(-3, 1)$ in the diagram are all ordered pairs.



Note: The origin is denoted $(0, 0)$.

- **<u>x-coordinate</u>:** The first number of an ordered pair which indicates the horizontal position of a point from the origin.
  Example: In the diagram above, -3, -2, 2, and 4 are x-coordinates.

- **y-coordinate:** The second number of an ordered pair which indicates the vertical position of a point from the origin.
  Example: In the diagram above, $-3, -2, 1, 3$, and 0 are y-coordinates.

Distance Formula

- The **distance** d between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example: Find the distance between $(4, -2)$ and $(-5, 8)$.
Let $P_1 = (x_1, y_1) = (4, -2)$ and $P_2 = (x_2, y_2) = (-5, 8)$

$$d = \sqrt{(-5 - 4)^2 + (8 - (-2))^2}$$
$$d = \sqrt{(-9)^2 + (8 + 2)^2} = \sqrt{81 + (10)^2}$$
$$d = \sqrt{81 + 100} = \sqrt{181}$$

Midpoint Formula

- The **midpoint** (x_m, y_m) of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is given by:

$$(x_m, y_m) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Example: Find the midpoint of the line segment joining the points (5, 4) and (-9, 2).
Use (5, 4) for (x_1, y_1) and (-9, 2) for (x_2, y_2).

$$(x_m, y_m) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$
$$(x_m, y_m) = (\frac{5 + (-9)}{2}, \frac{4 + 2}{2})$$
$$(x_m, y_m) = \left(\frac{-4}{2}, \frac{6}{2}\right) = (-2, 3)$$

Graphing with Points, Intercepts, and Symmetry

Key Definitions

- **Graph of an Equation:** uses two variables (x and y) to show the infinite pairs of x and y in the xy-plane that make the equation true (ex: $y = x + 1$)
- **X-intercept**: the point where the line crosses the x-axis (y=0)
- **Y-intercept**: the point where the line crosses the y-axis (x=0)

- **Symmetric with respect to the x-axis**: if the graph has a point at (a,b), it will also have a point at (a,-b); changing the y-value of a point to negative y will not change the equation
- **Symmetric with respect to the y-axis**: if the graph has a point at (a,b), it will also have a point at (-a,b); changing the x-value of a point to negative x will not change the equation
- **Symmetric with respect to the origin**: if the graph has a point at (a,b), it will also have a point at (-a,-b); changing the x-value and y-value of a point to negative x and negative y will not change the equation

Graphing with Points

- Use the equation to make a table with pairs of x and y that make the equation true—pick some numbers to be x-values and solve for y to find pairs of x and y.
- Plot the points on the xy-plane.
- Connect the points with a line.

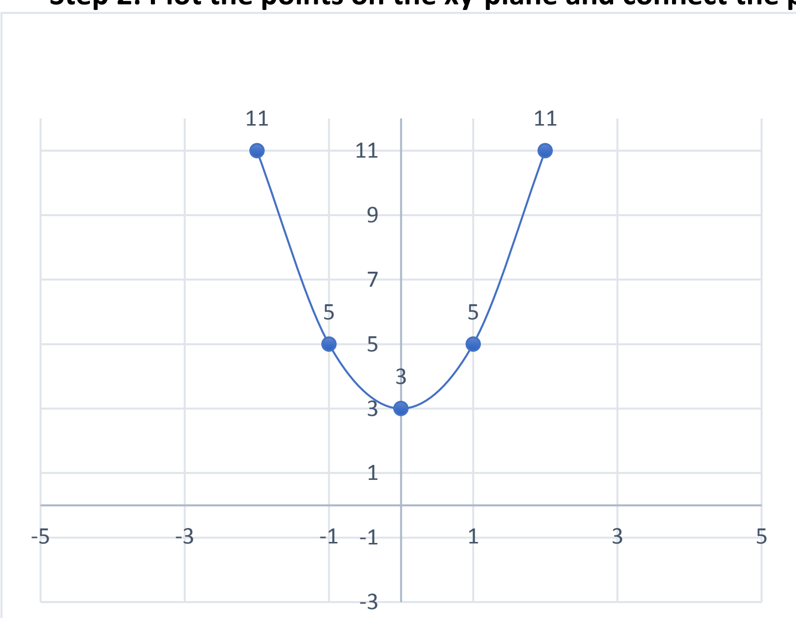
Example: Graph the equation $y = 2x^2 + 3$

Step 1: Make a table with x and y pairs.

Plug in x-values you pick to find y-values: $y = 2(0)^2 + 3 = 0 + 3 = 3$

x	y
0	3
1	5
-1	5
2	11
-2	11

Step 2: Plot the points on the xy-plane and connect the points with a line.



Finding Intercepts

- **To Find an X-intercept:** make y equal to zero and solve for x.
- **To Find a Y-intercept:** make x equal to zero and solve for y.

Example: Find the x-intercepts and y-intercepts (if any) of this graph: $y = x^2 - 4$

Step 1: Check for x-intercepts by making $\mathbf{y = 0}$.

$$0 = x^2 + 4$$
$$-4 = x^2$$

There are no real solutions for x, so this graph has no x-intercepts.

Step 2: Check for y-intercepts by making $\boldsymbol{x = 0}$.

$$y = (0)^2 - 4 = -4$$

The y-intercept is (0,-4).

Finding Symmetry

- **When testing for symmetry, a graph can have no symmetry, one kind of symmetry, or all three kinds of symmetry.**
- **Test for Symmetry about the x-axis:** Using the equation of the line, replace y with –y. If the equation remains the same, the graph is symmetric about the x-axis.
- **Test for Symmetry about the y-axis:** Using the equation of the line, replace x with –x. If the equation stays the same, the graph is symmetric about the y-axis.
- **Test for Symmetry about the origin:** Using the equation of the line, replace both x and y with –x and –y, respectively. If the equation stays the same, the graph is symmetric about the origin.

Example: Determine whether the graph is symmetric with respect to the x-axis, y-axis, or origin: $y = x^3 + x$.

Step 1: Check for symmetry about the x-axis.

$$(-y) = x^3 + x$$
$$-y = x^3 + x$$

Since the equation does not remain the same, the graph is not symmetric with respect to the x-axis.

Step 2: Check for symmetry about the y-axis.

$$y = (-x)^3 + (-x)$$
$$y = -x^3 - x$$

Since the equation does not remain the same, the graph is not symmetric with respect to the y-axis.

Step 3: Check for symmetry about the origin.

$$(-y) = (-x)^3 + (-x)$$
$$-y = -x^3 - x$$
$$y = x^3 + x$$

Since the equation does remain the same, the graph is symmetric with respect to the origin.

Graphing with Intercepts and Symmetry

- To graph an equation using intercepts and symmetry, begin by finding the intercepts and graphing those. Then test for the three kinds of symmetry, and use any symmetry found to graph enough points to sketch the graph.

  Example: Plot the graph of this equation. $x^2 - y^2 = 16$

  Step 1: Find the x-intercepts.

$$x^2 - (0)^2 = 16$$
$$x^2 = 16$$
$$x = \pm 4$$

The two x-intercepts are (4,0) and (-4,0).

Step 2: Find y-intercepts.

$$(0)^2 - y^2 = 16$$
$$-y^2 = 16$$
$$y^2 = -16$$

There are no real solutions for y, so this graph has no y-intercepts.

Step 3: Test for symmetry about the x-axis.

$$x^2 - (-y)^2 = 16$$
$$x^2 - y^2 = 16$$

Since the equation remains the same, the graph is symmetric about the x-axis.

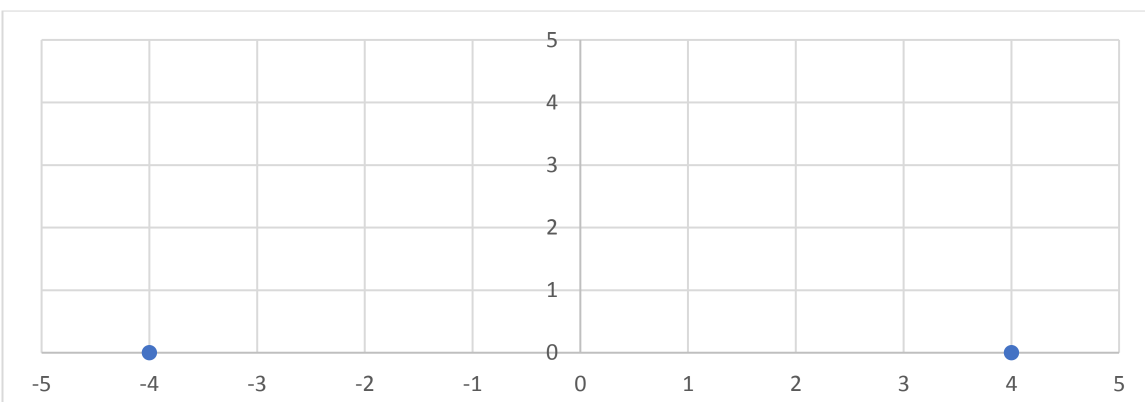
$$(-x)^2 - y^2 = 16$$
$$x^2 - y^2 = 16$$

Since the equation remains the same, the graph is symmetric about the y-axis.

$$(-x)^2 - (-y)^2 = 16$$
$$x^2 - y^2 = 16$$

Since the equation remains the same, the graph is symmetric about the origin.

Step 4: Plot the intercepts on the graph.



Step 5: Find other points using symmetry. Plot them on the graph, and sketch the graph.

Since $5^2 - 3^2 = 25 - 9 = 16$, (5,3) is a point on the graph.

Due to symmetry, these points are also on the graph: (-5,3), (-5,-3), (5,-3).

Plot these points and sketch the line. Since there is no y-intercept (from step 2), do not connect the two parts of the graph. Also, keep in mind that the graph should be symmetric about the y-axis, x-axis, and origin.