Functions

Key Definitions

- **Relation:** a correspondence between two sets where each element in the first set, called the **domain**, corresponds to *at least one element* in the second set, called the **range**.
- **Function:** a correspondence between two sets where each element in the first set, called the **domain**, corresponds to *exactly one element* in the second set, called the **range**.
- **Vertical Line Test:** given the graph of an equation, if any vertical line that can be drawn intersects the graph at no more than one point, the equation defines y as a function of x
- **Dependent Variable:** a variable that depends on the value of another variable selected (y is often used as a dependent variable)
- **Independent Variable:** a variable that can be any number in the domain (x is often used as an independent variable)
- **Function Notation:** "x" is the input or independent variable from the domain; "f" is the name of the function; "y" or "f(x)" (said "f of x") is the output or dependent variable which is in the range; "f(x) = 2x+5" is an example of the actual equation.

Functions

- **Example 1:** Determine whether the following relation is a function.

$$\{(-4,5),(-1,0),(2,3),(3,2)\}$$

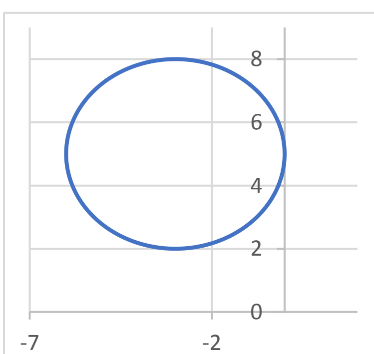
No x-values are repeated. Therefore, each element in the first set (x-values) corresponds to exactly one element in the second set (y-values), and the relation is a function.

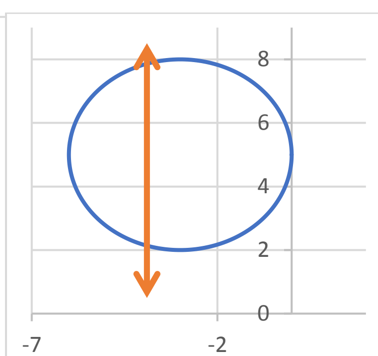
- **Example 2:** Determine whether the following relation is a function.

$$\{(-4,5),(-1,0),(2,3),(2,5)\}$$

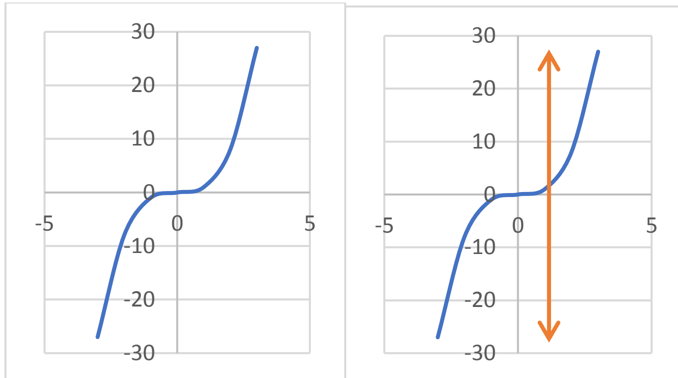
Since the x-value of 2 corresponds to two different y-values 3 and 5, this relation is not a function.

- Although functions are defined by equations, it is important to recognize that not all equations are functions.
- **Example 3**: Use the **vertical line test** to determine whether the graphs of equations define functions of x.





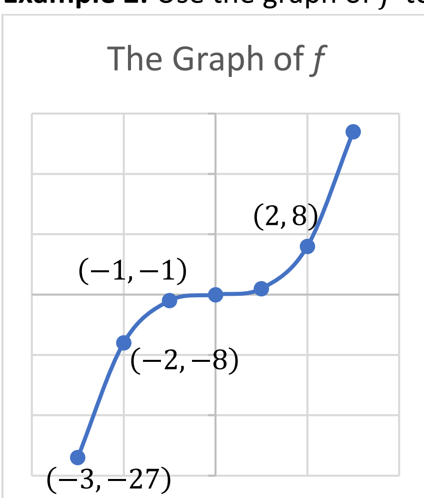
For this graph, many vertical lines can be drawn on the graph that will intersect the graph of the equation at two points. Therefore, this graph does not represent a function.



Any vertical line drawn will intersect the graph of the equation only once. Therefore, this equation represents a function.

Evaluating Functions

- **Example 1:** Given the function $f(x) = 3x^2 - x + 2$, find $f(-2)$.
  Step 1: Replace all x-values with (-2).
  $$f(-2) = 3(-2)^2 - (-2) + 2$$
  Step 2: Simplify the right side of the equation.
  $$f(-2) = 3(4) - (-2) + 2$$
  $$f(-2) = 12 + 2 + 2$$
  $$f(-2) = 16$$
- **Example 2:** Use the graph of f to find $f(-1)$, $f(-3)$, and $f(2)$.



  For $f(-1)$: The value $x = -1$ corresponds to the value $y = -1$. $f(-1) = -1$
  For $f(-3)$: The value $x = -3$ corresponds to the value $y = -27$. $f(-3) = -27$
  For $f(2)$: The value $x = 2$ corresponds to the value $y = 8$. $f(2) = 8$
- **Example 3:** Given the function $f(x) = 3x^2 + x$, evaluate $f(x - 1)$ and simplify if possible.
  Step 1: Replace all x-values with $(x - 1)$.
  $$f(x - 1) = 3(x - 1)^2 + (x - 1)$$

Step 2: Simplify the right side of the equation.

$$f(x-1) = 3(x^2 - 2x + 1) + (x - 1)$$
$$f(x-1) = 3x^2 - 6x + 3 + x - 1$$
$$f(x-1) = 3x^2 - 5x + 2$$

- **Example 4:** Given the function $f(x) = 3x^2 + x$, evaluate $f(x+1)$ and $f(x) + f(1)$ and simplify if possible.

  Step 1: First find $f(x+1)$. Replace all x-values with $(x+1)$.

  $$f(x+1) = 3(x+1)^2 + (x+1)$$

  Step 2: Simplify the right side of the equation.

  $$f(x+1) = 3(x^2 + 2x + 1) + (x + 1)$$
  $$f(x+1) = 3x^2 + 6x + 3 + x + 1$$
  $$f(x+1) = 3x^2 + 7x + 4$$

  Step 3: Find $f(1)$. Replace all x-values with (1).

  $$f(1) = 3(1)^2 + (1)$$

  Step 4: Simplify the right side of the equation.

  $$f(1) = 3(1) + (1)$$
  $$f(1) = 3 + 1$$
  $$f(1) = 4$$

  Step 5: Add $f(x) + f(1)$.

  $$f(x) + f(1) = (3x^2 + x) + (4)$$
  $$f(x) + f(1) = 3x^2 + x + 4$$

Determining the Domain of a Function

- When determining the domain of a function, ask "what values can x be?" or "what can x not be?" to determine the restrictions on x.
- **Example 1**: State the domain of the function: $F(x) = \frac{5}{x^2-36}$

  Step 1: Determine any restrictions on the values of x.

  $$x^2 - 36 \neq 0$$
  $$x^2 \neq 36 \; or \; x \neq \pm 6$$

  Step 2: Write the domain in interval notation.

  $$(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$$

- **Example 2**: State the domain of the function: $F(x) = \sqrt[4]{10 - 3x}$

  Step 1: Determine any restrictions on the values of x.

  $$10 - 3x \geq 0$$
  $$10 \geq 3x$$
  $$x \leq \frac{10}{3}$$

  Step 2: Write the domain in interval notation.

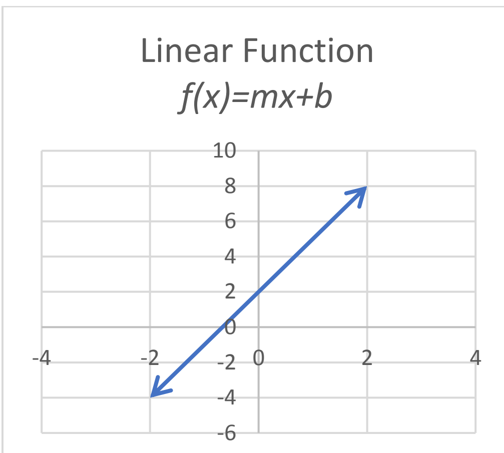
  $$\left(-\infty, \frac{10}{3}\right]$$

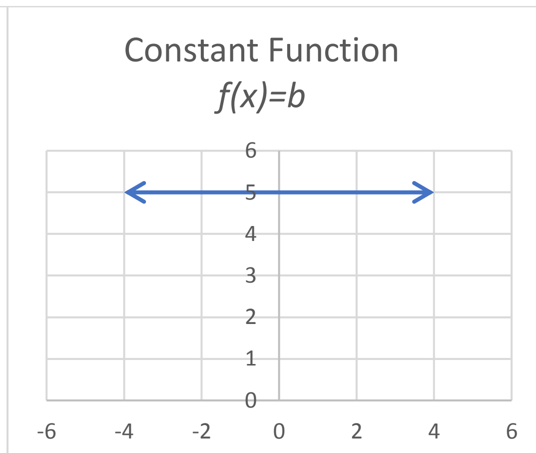
- **Example 3**: State the domain of the function: $F(x) = \sqrt[3]{1-x}$
  Step 1: Determine any restrictions on the values of x.
  There are no restrictions on the value of x, so the domain is all real numbers.
  Step 2: Write the domain in interval notation.

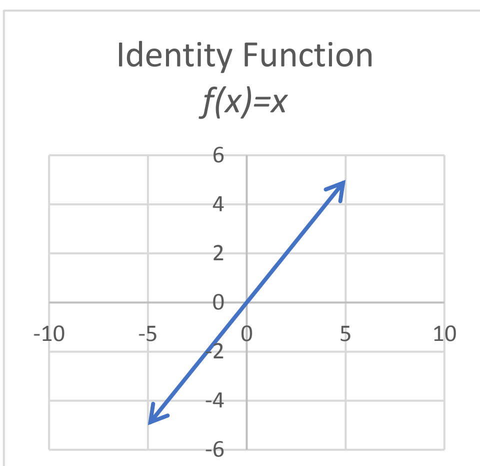
$$(-\infty, \infty)$$

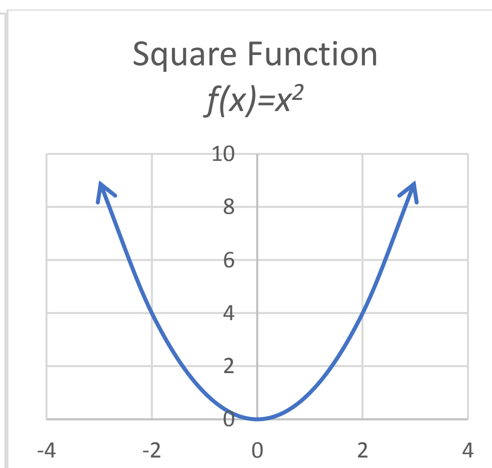
Graphs of Functions

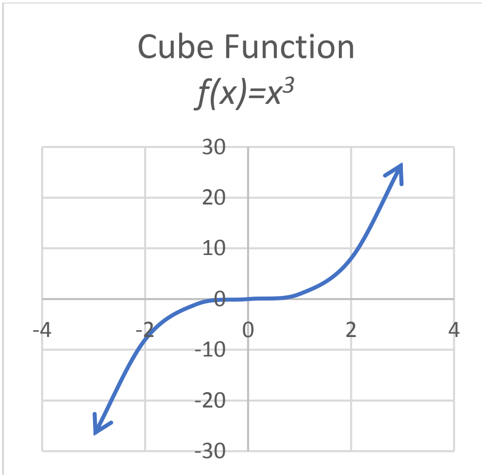
Common Functions

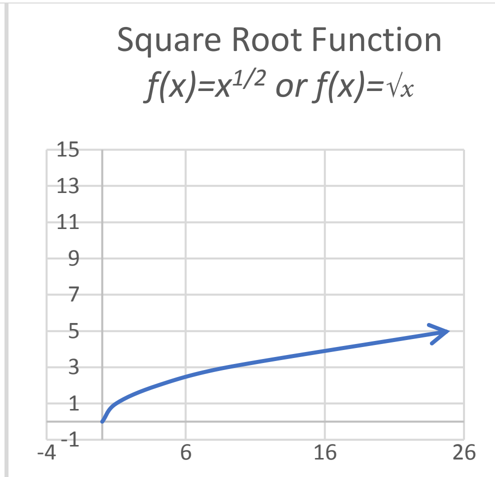


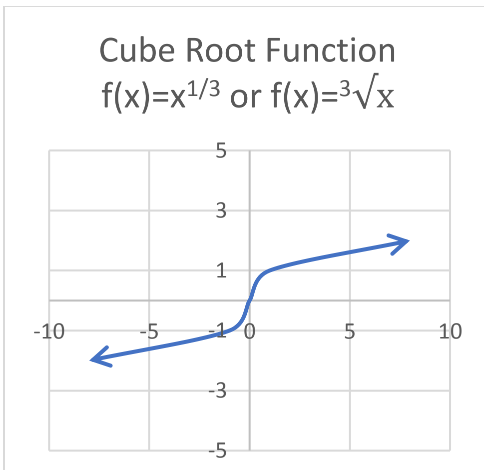


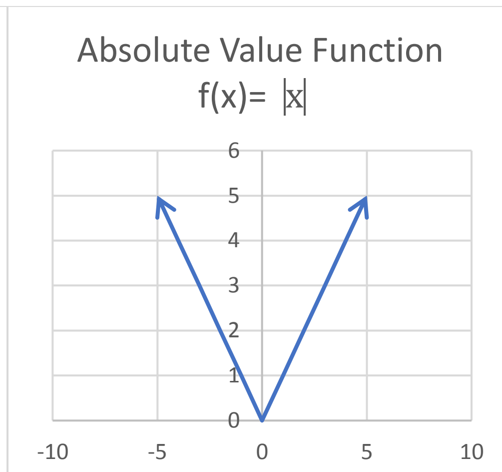


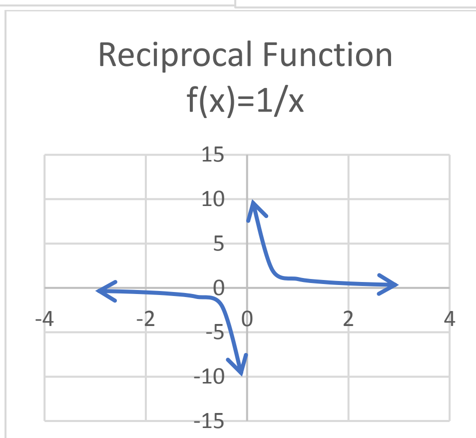












Even and Odd Functions

Even and Odd Functions		
Function	Symmetric with Respect to	Replacing x with -x
Even	y-axis or vertical axis	$f(-x) = f(x)$
Odd	Origin	$f(-x) = -f(x)$

- **Example:** Determine whether the functions are even, odd, or neither by replacing x with –x in each function.

$$1.\, f(x) = x^4 - 2, \qquad 2.\, h(x) = x^5 - x^3, \qquad 3.\, g(x) = x$$

1. Replace x with –x in $f(x) = x^4 - 2$.

$$f(-x) = (-x)^4 - 2$$
$$f(-x) = x^4 - 2 = f(x)$$

Since the power 4 is even, $(-x)^4 = x^4$, and the function is even because $f(-x) = f(x)$.

2. Replace x with –x in $h(x) = x^5 - x^3$.

$$h(-x) = (-x)^5 - (-x)^3$$

Since the powers 5 and 3 are odd, $(-x)^5 = -x^5$ and $(-x)^3 = -x^3$.

$$h(-x) = -x^5 - -x^3 = -x^5 + x^3$$

Since $h(-x)$ is not $h(x)$ nor $-h(x)$, the function is neither even nor odd.

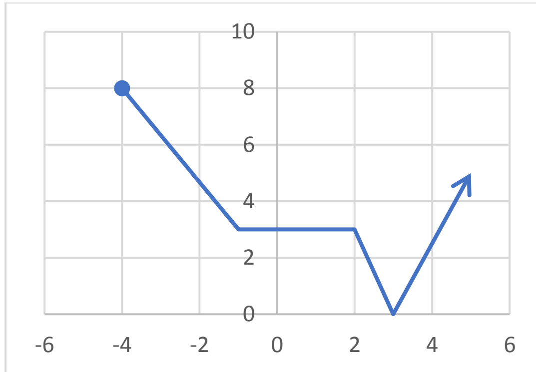
3. Replace x with –x in $g(x) = x$.

$$g(-x) = -x = -g(x)$$

Since $g(-x) = -g(x)$, the function is odd.

Increasing and Decreasing Functions

- A function f is **increasing** on an open interval *I* if for any x_1 and x_2 in *I*, where $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- A function f is **decreasing** on an open interval *I* if for any x_1 and x_2 in *I*, where $x_1 < x_2$, then $f(x_1) > f(x_2)$.
- A function f is **constant** on an open interval *I* if for any x_1 and x_2 in *I*, then $f(x_1) = f(x_2)$.
- The **domain** is the set of all x-values (from left to right) where the function is defined (all possible x-values).
- The **range** is the set of y-values (from bottom to top) that the graph of the function corresponds to (all possible y-values).
- Intervals of increasing or decreasing are defined on open intervals which means these intervals should be written in set notation using open notation (and parenthesis).
- **Example: Given the graph of a function, state the domain and range of the function; then find the intervals when the function is increasing, decreasing, or constant.**



The domain of the function is $[-4, \infty)$, and the range of the function is $[0, \infty)$.
Reading the graph from left to right, we see that that graph
Decreases from the point (-4, 8) to (-1, 3)
Is constant from the point (-1, 3) to (2, 3)
Decreases from the point (2, 3) to (3, 0)
Increases from the point (3, 0) on
Write the intervals using the x-coordinates:
The function is increasing on the interval $(3, \infty)$, decreasing on the interval $(-4, -1) \cup (2,3)$, and constant on the interval $(-1,2)$.

Average Rate of Change

- Let $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two distinct points, $(x_1 \neq x_2)$, on the graph of the function f. The **average rate of change** of f between x_1 and x_2 is given by:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- The expression $\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$, is called the **difference quotient**.
- **Example:** Find the average rate of change of $f(x) = x^2$ from: $x = -1\ to\ x = 0$ and $x = 0\ to\ x = 2$.
  Use the formula for the average rate of change

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

  Plug in the x-values to find the average rate of change from $x = -1\ to\ x = 0$:

$$\frac{f(0) - f(-1)}{0 - (-1)}$$

$$\frac{(0)^2 - (-1)^2}{0 - (-1)}$$

$$\frac{0-1}{0+1}=\frac{-1}{1}=-1$$

Plug in the x-values to find the average rate of change from $x = 0\ to\ x = 2$:

$$\frac{f(2)-\ f(0)}{2-0}$$

$$\frac{(2)^2-(0)^2}{2}$$

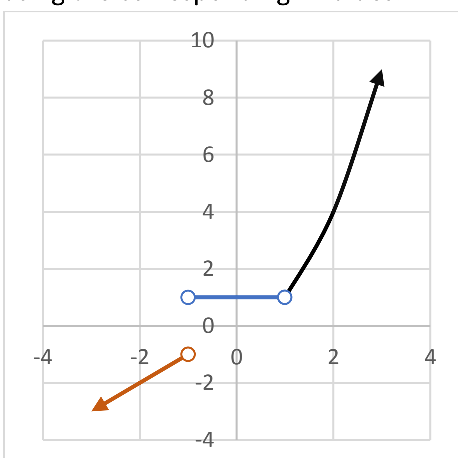
$$\frac{4-0}{2}=\frac{4}{2}=2$$

Piecewise-Defined Functions

- **Piecewise-Defined Functions:** functions that are defined in terms of pieces of other functions
- **Example 1:** Graph the piecewise-defined function, and state the domain, range, and intervals when the function is increasing, decreasing, or constant.

$$G(x)=\begin{cases} x, & x<-1 \\ 1, & -1\le x\le 1 \\ x^2, & x>1 \end{cases}$$

Graph each piece of the graph using the corresponding x-values.



Determine the domain and range of the function by looking at the possible x-values and y-values, resp.

The domain of the function is $(-\infty,\infty)$ since all x-values are possible.

The range of the function is $(-\infty,-1)\cup[1,\infty)$ since the function jumps from $y=-1$ to $y=1$ (and does not include $y=-1$).

Look at the function from left to right to determine if it is increasing, decreasing, or constant on certain intervals.

The function is constant on the interval $(-1,1)$.

The function is increasing on the interval $(-\infty, -1) \cup (1, \infty)$.
The function is never decreasing.

One-to-One and Inverse Functions

Key Definitions

- A function $f(x)$ is **one-to-one** if no two inputs map to the same output; each input must have a different output.
- **Horizontal Line Test:** If every horizontal line intersects the graph of a function in at most one point, then the function is a one-to-one function.
- The **inverse function $\boldsymbol{f^{-1}(x)}$** maps the output back to the input of the function f.
- The domain of f is the range of $f^{-1}(x)$. The range of f is the domain of $f^{-1}(x)$.
- $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$
- If the point (a, b) is on the graph of a function, then the point (b, a) is on the graph of its inverse.

One-to-One Functions

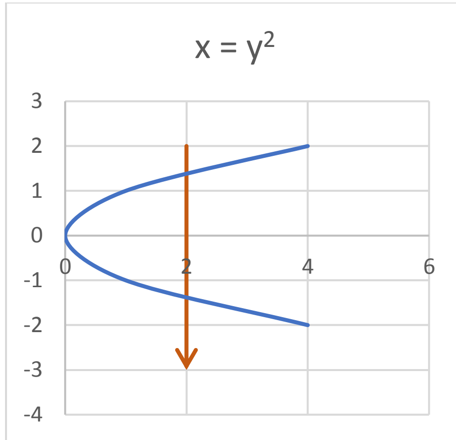
- **Example 1:** For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function.

$$f = \{(4,7), (5,8), (6,8)\}$$
$$g = \{(4,7), (5,8), (5,9)\}$$
$$h = \{(4,7), (5,8), (6,9)\}$$

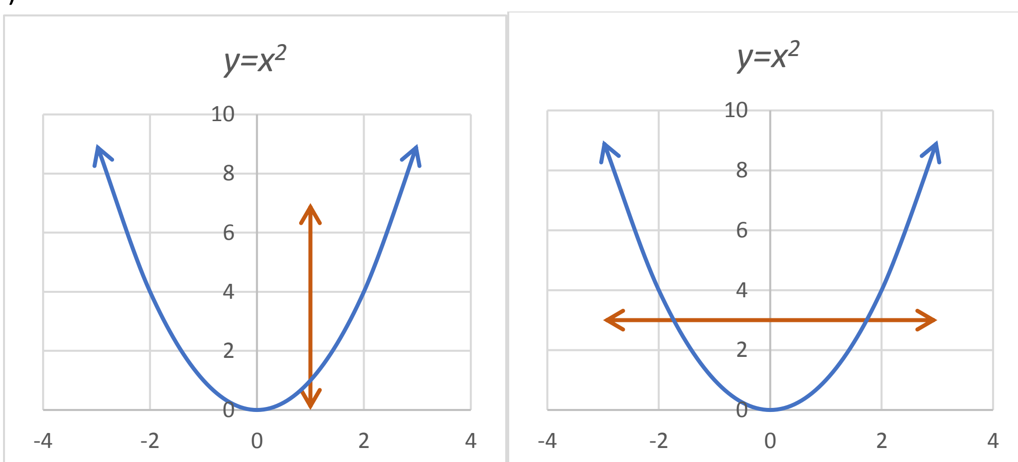
For $f = \{(4,7), (5,8), (6,8)\}$, no x-values are repeated. Therefore, each element in the first set (x-values) corresponds to exactly one element in the second set (y-values), and the relation is a function. However, it is not one-to-one because two inputs (5 and 6) map to the same output (8).
For $g = \{(4,7), (5,8), (5,9)\}$, since the x-value of 5 corresponds to two different y-values 8 and 9, this relation is not a function.
For $h = \{(4,7), (5,8), (6,9)\}$, no x-values are repeated. Therefore, each element in the first set (x-values) corresponds to exactly one element in the second set (y-values), and the relation is a function. It is one-to-one because each input has a different output.

- **Example 2:** For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function. Assume that x is the independent variable and y is the dependent variable.

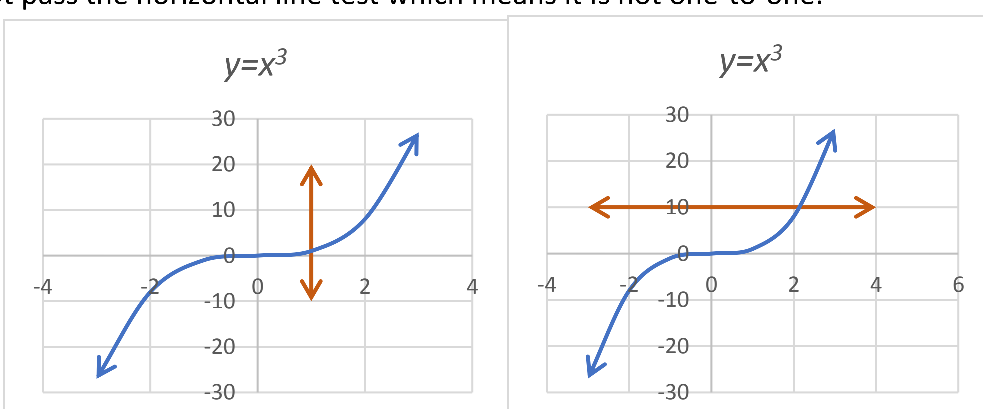
$$x = y^2$$
$$y = x^2$$
$$y = x^3$$



This relation (in blue) is not a function because it does not pass the vertical line test (in orange).



This relation is a function because it passes the vertical line test. However, the function does not pass the horizontal line test which means it is not one-to-one.



This relation is a function because it passes the vertical line test. It also passes the horizontal line test which means it is a one-to-one function.

Inverse Functions

- **Example 1:** Verify that $f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$ is the inverse of $f(x) = 3x + 2$.

Show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$

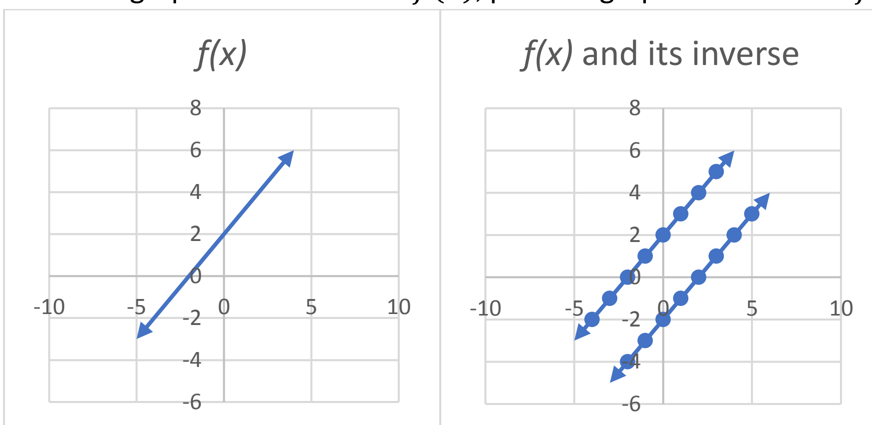
$$f^{-1}\big(f(x)\big) = \frac{1}{3}f(x) - \frac{2}{3}$$

$$f^{-1}\big(f(x)\big) = \frac{1}{3}(3x + 2) - \frac{2}{3} = x + \frac{2}{3} - \frac{2}{3} = x$$

$$f\big(f^{-1}(x)\big) = 3\,f^{-1}(x) + 2$$

$$f\big(f^{-1}(x)\big) = 3\left(\frac{1}{3}x - \frac{2}{3}\right) + 2 = x - 2 + 2 = x$$

- **Example 2:** Given the graph of the function $f(x)$, plot the graph of its inverse $f^{-1}(x)$.



Find the points of $f(x)$ and switch the x and y coordinate for each one to get the points for the inverse.

- **To find the inverse of a function:** Let $y = f(x)$. Then switch x and y. Then solve for y in terms of x. Then let $y = f^{-1}(x)$.
- **Example 3**: Find the inverse of $f(x) = 2x - 6$.

Let $y = f(x)$.

$$y = 2x - 6$$

Switch x and y.

$$x = 2y - 6$$

Solve for y.

$$x + 6 = 2y - 6 + 6 = 2y$$

$$\frac{x+6}{2} = \frac{2y}{2} = y$$

$$y = \frac{1}{2}x + 3$$

Let $y = f^{-1}(x)$.

$$f^{-1}(x) = \frac{1}{2}x + 3$$

- **Important: If the function fails the horizontal line test and is not one-to-one, its inverse does not exist.**