

# Probability

## Key Definitions

- **Probability:** The likelihood or chance of something happening (between 0 and 1).
- **Law of Large Numbers:** The more data you have, the more true to the probability of the outcome your experiment results will become.
- **Sample Space ( $S$ ):** A list of all the possible results in an experiment.
- **Event ( $E$ ):** The desired result of a probability experiment.
- **Unusual Event:** An event that has a very low likelihood of happening.
- **Impossible Event:** An event with probability 0.
- **Certain Event:** An event with probability 1.
- **Disjoint (Mutually Exclusive) Events:** Events are disjoint if they have no common outcomes (There is not intersection in the Venn diagram).
- **Complement of  $E$ :** Outcomes that occurred your sample space  $S$ , but are not outcomes of your event  $E$ .
- **Independent Events:** Events are independent if the outcome of the first experiment does not impact the probability of the second experiment.
- **Dependent Events:** Events are dependent if the outcome of the first event affects the probability of the second event.
- **Conditional Probability:** The probability of a dependent event occurring given the event that impacts the probability has already occurred.
- **Permutation:** A way to count how many ways something can be arranged when order matters.
- **Combination:** A way to count how many ways something can be arranged when order *does not* matter.

## Probability Rules

- **Rules of Probabilities:**
  - Any probability  $P(E)$  must satisfy:  $0 \leq P(E) \leq 1$ . This means the likelihood of something occurring cannot be less than 0% or more than 100%
  - Any probability  $P(E)$  must satisfy:  $P(e_1) + P(e_2) + \dots + P(e_n) = 1$ . This means when all of the probabilities of every possible outcome are added together, they must equal 1.

## Finding Probabilities Using the Empirical Approach

- **Approximating Probabilities using the Empirical Approach:**  
The empirical approach requires us to look at the results of an evidence-based probability experiment. This is why we use relative frequency. Relative frequency is the outcome based of an experiment that has been performed numerous amounts of times. This operates the same way as the relative frequency in chapter 2. The formula is the following:

$$P(E) \approx \text{relative frequency of } E = \frac{\text{frequency of } E}{\# \text{ of trials of experiment}}$$

\*Notice: as we increase our number of trials, our data will become more accurate (by the Law of Large Numbers).

- **Example of Approximating Probabilities using the Empirical Approach:**

A hotel housed people in 117 rooms on a given night. 98 of those rooms reported that they were satisfied with their stay that night. Using the Empirical Method:

$$P(E) = \frac{98}{117} \approx 0.8376$$

Based on this result, we can approximate that out of every 100 people staying in the hotel, about 84 will be satisfied with their stay. If this process were to be repeated for multiple nights, our data would become more and more accurate.

## Finding Probabilities Using the Classical Method

- **Computing Probabilities using the Classical Method:**

If  $n$  is the number of equally likely outcomes and  $m$  is the number of ways the event  $E$  can occur, then

$$P(E) = \frac{\# \text{ of ways } E \text{ can occur}}{\# \text{ of possible outcomes}} = \frac{m}{n}$$

If  $S$  were the sample space of the experiment:

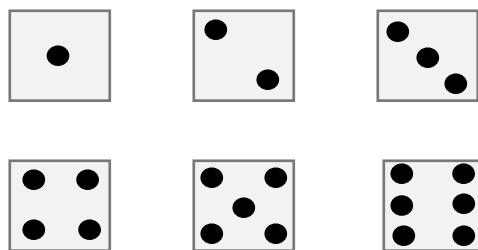
$$P(E) = \frac{N(E)}{N(S)}$$

Where  $N(E)$  = the number of outcomes in  $E$  and  $N(S)$  = the number of possible outcomes within the sample space.

- **Example of Computing Probabilities using the Classical Method:**

Consider all the different ways to roll a one die. What is the likelihood of rolling a two?

First, we need to determine the outcomes that are possible when we roll a die.



Since there are 6 different way to role a die (our sample space) and only 1 way to roll a two (our event):

$$N(S) = 6 \quad \text{and} \quad N(E) = 1$$

$$\text{So, } P(E) = \frac{1}{6}$$

## The Addition Rules

- **The Addition Rule for Disjoint Events:**

When we have two disjoint events happening at the same time and we would like to know the outcome of the first event or the second event happening, we take the probability of the first event and add it to the probability of the second event. This is because both events are mutually exclusive and we are looking for one or the other. The formula is as follows:

$$P(E \text{ or } F) = P(E) + P(F)$$

Where  $E$  and  $F$  are two mutually exclusive events.

- **Example of the Addition Rule for Disjoint Events:**

Say you are given a jar of jellybeans where there are 17 yellow, 13 blue, 18 pink, 20 red, and 8 green for a total of 74 jelly beans. What is the probability of randomly selecting a red jelly bean or a green jelly bean?

First, we need to find the probability of getting a red jelly bean and the probability of getting a green jelly bean.

$$P(E_{\text{Red}}) = \frac{20}{74} = \frac{10}{37} \quad \text{and} \quad P(E_{\text{Green}}) = \frac{8}{74} = \frac{4}{37}$$

Then, we need to add the probabilities together.

$$P(E_{\text{Red}}) + P(E_{\text{Green}}) = \frac{10}{37} + \frac{4}{37} = \frac{14}{37} \approx 0.3784$$

- **The General Addition Rule:**

When we have two events that are not mutually exclusive, this means the outcome can happen in both events. We are still looking at the first event or the second event happening in this addition rule. So, we still need to add the two probabilities together, but we have the same outcome being repeated in both

events. This means we have counted twice for the outcome when we should have only counted once. So, we find which outcome is shared and subtract it from our previous addition to get rid of the extra counting of the one outcome. The formula for this is as follows:

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

- **Example of the General Addition Rule:**

What is the likelihood of drawing a heart or a 6 from a standard card deck of 52 cards?

First, we need to know how many hearts are in a standard deck of cards and how many sixes are in a standard deck of cards. Since hearts is a suit and there are 4 suits in a standard deck of cards, we find that there are  $\frac{52}{4} = 13$  hearts in a deck of cards. Next, there is one six in each suit for our standard deck of cards, and there is 4 suits in a standard deck of cards. This means there are 4 sixes in a standard deck of cards.

Next, we need to find the individual probability for each event.

$$P(E_{\text{Heart}}) = \frac{13}{52} \quad \text{and} \quad P(E_{\text{Six}}) = \frac{4}{52}$$

Now, we need to find if there is an outcome that happens in both events. Since there is a 6 of hearts in a standard deck of cards, this means we have found the probability of getting a 6 of hearts twice. This means our event are not mutually exclusive and we need to find the probability of getting a hearts and a 6. Since there is only one 6 that is part of the suit of hearts, we have the following probability:

$$P(E_{\text{Heart and Six}}) = \frac{1}{52}$$

Finally, we can take the information we have found and put it into our formula. This gives us the following:

$$P(E_{\text{Heart or Six}}) = P(E_{\text{Heart}}) + P(E_{\text{Six}}) - P(E_{\text{Heart and Six}}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13} \approx 0.3077$$

## The Complement Rule

- **The Complement Rule:**

When we are asked the probability of an event in a sample space and we know the probability of the other event in the sample space outcome, we can find the probability for the outcome we are looking for simply. We take the known probability and subtract it from 1 (this is because all probability adds up to equal 1). This is shown in the following formula:

$$P(E^c) = 1 - P(E)$$

Where  $E^c$  is the complement of  $E$

- **Example of the Complement Rule:**

Say there is a 0.71 chance of a basketball player making a free shot and succeeding. What is the probability that the basketball player will miss?

$$P(E^c) = 1 - 0.71 = 0.29$$

## Independence and the Multiplication Rule

- **The Multiplication Rule for Independent Events:**

When we are looking for the probability of 2 events happening together at the same time, we cannot use the addition rule. Since the outcomes must happen together, we take the probability of both events and multiply them together. This is represented by the following formula:

$$P(E \text{ and } F) = P(E) \cdot P(F)$$

In the case of more than 2 events, we can use the same formula, but we expand it to show all the events. This is represented by the following formula:

$$P(E_1 \text{ and } E_2 \text{ and } E_3 \dots \text{ and } E_n) = P(E_1) \cdot P(E_2) \cdot P(E_3) \cdot \dots \cdot P(E_n)$$

- **Example of the Multiplication Rule for Independent Events:**

A Regent student needs to finish getting dressed for class by picking out a pair of socks, a shirt, and a pair of pants. The student is very neat and has all their socks rolled into pairs in their top dresser drawer. They have 10 pairs of socks: 3 black pairs, 3 blue pairs, 2 grey pairs, and a white pair. The student also has all their t-shirts folded neatly in their second dresser drawer. The student has 12 t-shirts: 4 Regent shirts, 3 shirts with other graphics, 3 solid colored shirts, and 2 striped shirts. The student keeps their pants folded in the bottom drawer of their dresser. They have 7 pairs of pants: 3 pairs of khakis, 2 pairs of sweatpants, and 2 pairs of jeans. What is the probability that the student selects a grey pair of socks, a solid colored shirt, and a pair jeans to wear to class?

First, we need to find the probabilities of each event.

$$P(\text{Grey Socks}) = \frac{2}{10}, P(\text{Solid T-Shirt}) = \frac{3}{12}, P(\text{Jeans}) = \frac{2}{7}$$

Next, we plug it into our formula and solve.

$$P(\text{Grey Socks}) \cdot P(\text{Solid T-Shirt}) \cdot P(\text{Jeans}) = \frac{2}{10} \cdot \frac{3}{12} \cdot \frac{2}{7} = \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{2}{7} = \frac{2}{140} = \frac{1}{70} \approx 0.01429$$

## Conditional Probability

- **The Conditional Probability Rule:**

When we are looking for the probability of an event given the probability of a different event, we use our conditional probability rule. This can also be stated as the likelihood of an event  $F$  occurring after event  $E$  has occurred. It is represented by the following formula:

$$P(F | E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{N(E \text{ and } F)}{N(E)}$$

- **Example of the Conditional Probability Rule:**

The Regent student from above did a load of laundry which included all 10 pairs of his socks. Before the student can roll them into pairs, they must first pull the socks out of their laundry basket, one by one. What is the probability that the student selects a grey sock from his laundry basket given that the first sock he selected was grey? (Remember: The student has 2 pairs of grey socks)

**Note:** The Multiplication Rule for independent events cannot be used here because these events are not independent. Unlike the first problem involving this Regent student, which involved him choosing one item from 3 different drawers, this problem has him choosing 2 items from the same laundry basket. What he selects first will affect what is left for him to select second.

$$P(\text{2nd Grey}|\text{1st Grey}) = \frac{P(\text{1st Grey and 2nd Grey})}{P(\text{1st Grey})}$$

First, we need to find the probability of choosing a grey sock. Then, we need to find the probability of choosing a second grey sock from the remaining socks. (Remember that this means we have one less grey sock to choose from, as well as one less sock to choose from in total.)

$$P(\text{1st Grey and 2nd Grey}) = \frac{4}{20} \cdot \frac{3}{19} = \frac{1}{5} \cdot \frac{3}{19} = \frac{3}{95}$$

The first step gives us the probability for our denominator.

$$P(\text{1st Grey}) = \frac{4}{20} = \frac{1}{5}$$

Finally, we plug the two probabilities into the original fraction to get the probability of choosing a grey sock the second time given that the first sock chosen was also grey.

$$P(\text{2nd Grey}|\text{1st Grey}) = \frac{\frac{3}{95}}{\frac{1}{5}} = \frac{3 \cdot 5}{95} = \frac{3}{19}$$

## Counting Techniques

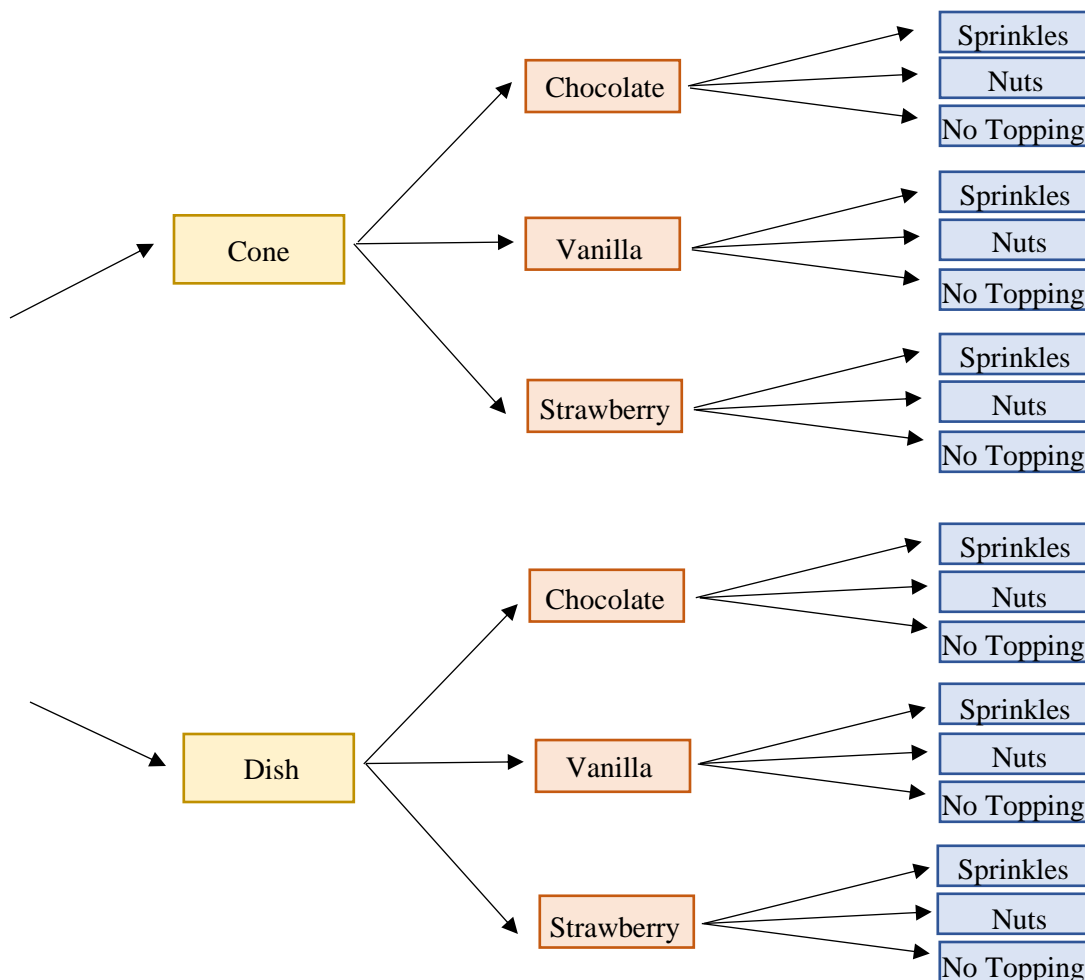
- **The Multiplication Rule of Counting:**

If you are presented with a sequence of choice where the first choice has  $p$  options, the second choice has  $q$  choices and the third has  $r$  choices, and so on, then use

$$p \cdot q \cdot r$$

to determine how many different ways the selection can be done.

- **Example of the Multiplication Rule of Counting:**



At an ice cream parlor, you could either get your ice cream in a cone or a dish. From there, you could either order chocolate, vanilla, or strawberry ice cream. Finally, you have the option of putting sprinkles, nuts, or no topping on your ice cream. How many different ways could you order a serving of ice cream?

Since there are 2 options for what the ice cream comes in, 3 for what kind of ice cream you order, and 3 for what goes on top of your ice cream, there are 18 different ways to order ice cream from this parlor.

$$2 \cdot 3 \cdot 3 = 18$$

- **Factorials:**

Factorials is a way we can count things. Factorial means that we have  $n$  objects, and we are multiplying down until there are no objects left. It is represented by the operation symbol "!". This is shown in the formula down below:

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1$$

- **Example of a Factorial:**

To find 7! Multiply all integers (whole numbers) from 1 to 7.

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

- **Number of Permutations of  $n$  Distinct Objects Taken  $r$  at a Time:**

You can arrange  $n$  distinct objects (think about a group of people in a room) in  $r$  different ways (think taking  $r$  of them out of a room one at a time) without replacement (don't put them back in the room after you start) using the formula below. Permutations are used when the order matters (think the first person gets first prize and so on).

$${}_n P_r = \frac{n!}{(n-r)!}$$

- **Example of a Permutation:**

At the Regent University Spring Fest, a first, second, and third prize were awarded for best painting in a timed painting contest from among the 24 people who participated. How many ways could these prizes be awarded?

$$n = 24 \text{ and } r = 3$$

$${}_{24} P_3 = \frac{24!}{(24-3)!} = \frac{24!}{21!} = \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot \dots \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{21 \cdot 20 \cdot 19 \cdot \dots \cdot 3 \cdot 2 \cdot 1} = \frac{24 \cdot 23 \cdot 22}{1} = 12,144$$

**Note:** Do not multiply all of the numbers in the numerator and denominator and then divide. Cancel all common factors before multiplying. Notice that we were able to cancel all numbers from 1 to 21 in the above example because they appeared in both the numerator and denominator.

- **Number of Combinations of  $n$  Distinct Objects Taken  $r$  at a Time:**

You can arrange  $n$  distinct objects (think about a group of people in a room) in  $r$  different ways (think taking  $r$  of them out of a room one at a time) without replacement (don't put them back in the room after you start) using the formula below. Combinations are used when the order does NOT matter (think picking people for a team).

$${}_n C_r = \frac{n!}{(n-r)! \cdot r!}$$

- **Example of a Combination:**

A GENE 100 class composed of 48 students were broken into groups of 4 to discuss a topic from their readings. How many different possible combinations were available for the first group?

$$n = 48 \text{ and } r = 4$$

$$\begin{aligned} {}_n C_r &= \frac{48!}{(48-4)! \cdot 4!} = \frac{48!}{44! \cdot 4!} = \frac{48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot \dots \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(44 \cdot 43 \cdot 42 \cdot \dots \cdot 3 \cdot 2 \cdot 1) \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{48 \cdot 47 \cdot 46 \cdot 45}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{12 \cdot 47 \cdot 23 \cdot 15}{1} = 194,580 \end{aligned}$$



**Note:** Even if you cannot completely cancel all of the numbers in the denominator, you can still cancel them when numbers from the numerator have them as a factor. In the above example,

$$\frac{48}{4} = 12, \quad \frac{46}{2} = 23, \quad \text{and} \quad \frac{45}{3} = 15$$

- **Number of Permutations of  $n$  Non-Distinct Objects Taken  $r$  at a Time:**

You can arrange  $n$  objects (think ping pong balls in a bucket) that are divided into  $n_k$  distinct groups (think different colors of ping pong balls) using the formula below. This method is used when each object cannot be distinguished from the others in its group (all ping pong balls of one color look the same).

$$\frac{n!}{n_1! \cdot \cdots \cdot n_k!}$$

**Note:** The number of items in each group must be equal to  $n$  when added together. The subscripts 1 to  $k$  denote the distinct nature of the subgroup.

- **Example of Permutations of Non-Distinct Objects:**

A Regent University SOAR activity required the event leaders to arrange new students to start the activity. It does not matter if teams get mixed in the process; however, new students were indistinguishable to the event leaders except for their brightly colored team shirts. Members of 3 teams were present: 5 from the lime green team, 4 from the electric blue team, and 3 from the lightning yellow team. If 12 students in total participated in this activity, how many ways can the new students be arranged?

$$\frac{12!}{5! \cdot 4! \cdot 3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = \frac{3 \cdot 11 \cdot 5 \cdot 4 \cdot 7 \cdot 6}{1} = 27,720$$

## Finding Probabilities Using Counting Techniques

- **Obtaining Probabilities Using Permutations and Combinations:**

Sometimes obtaining the probability of an event requires the use of permutations or combinations. It is more likely that you will need to use combinations than permutations to complete this form of probability problem. To complete this problem type, follow the following steps:

1. Find the number of ways in which your desired event can be chosen (find all arrangements that satisfy the desired result):  $N(E)$
2. Find the total number events that can occur (find the total number of possible arrangements):  $N(S)$
3. Divide the number found in step one by the number found in step two to find the probability of the desired event occurring:  $P(E)$

**Note:** The result should be a decimal between 0 and 1.

- **Example of Obtaining Probabilities Using Combinations**

The class of 48 GENE 100 students from earlier broke into groups of 4 to facilitate class discussion. What is the probability that the first group was composed of 3 boys and 1 girl if there were 20 boys and 28 girls in the class.

First, set up the fraction with the information given in the problem. This problem requires that we find three combinations to find the probability. The general form is as follows:

$$\frac{N(E)}{N(S)} = \frac{{}_{20}C_3 \cdot {}_{28}C_1}{{}_{48}C_4}$$

**Note:** The  $n$ 's and  $r$ 's from the numerator of your fraction need to equal the  $n$  and  $r$  in the denominator when added, respectively.

Second, find each of the combinations individually. Luckily, we already know that  ${}_{48}C_4 = 194,580$ . The other two are completed below:

$${}_{20}C_3 = \frac{20!}{(20-3)! \cdot 3!} = \frac{20!}{17! \cdot 3!} = \frac{20 \cdot 19 \cdot 18 \cdot \dots \cdot 3 \cdot 2 \cdot 1}{(17 \cdot 16 \cdot \dots \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = \frac{10 \cdot 19 \cdot 6}{1} = 1,140$$

$${}_{28}C_1 = \frac{28!}{(28-1)! \cdot 1!} = \frac{28!}{27! \cdot 1!} = \frac{28 \cdot 27 \cdot 26 \cdot \dots \cdot 3 \cdot 2 \cdot 1}{(27 \cdot 26 \cdot \dots \cdot 2 \cdot 1) \cdot 1} = \frac{28}{1} = 28$$

Third, multiply the two combinations from the numerator, then divide them by the combination from the denominator.

$$\frac{1,140 \cdot 28}{194,580} = \frac{31,920}{194,580} \approx .16405$$

There is approximately a 16% chance that the first group would consist of 3 boys and 1 girl given the class demographics.

## Symbol Guide

Chapter Title Symbols		
Term	Symbol	Use
Probability of an Event	$P()$	Chance of an Outcome
Sample Space	$S$	All Outcomes
Event	$E$	A Desired Outcome
Number in a Set	$N()$	Number of Outcomes
Complement of an Event	$E^c$	Outcomes NOT Part of $E$
Permutation	${}_nP_r$	When Order Matters
Combination	${}_nC_r$	When Order does NOT Matter