Set Theory

Key Definitions

- **Set**: A set is a collection of things (numbers, pencils, cows, etc.). Sets are denoted by capital letters and braces.
- **Element**: An element of a set is one of the things that belong to the set (3 is an element of the set of all odd numbers). Elements are denoted by lower case letters.
- **Universal Set**: The universal set is the set of all elements being considered (the universal set of classes at Regent University is every course in the course catalogue). The universal set is denoted by "U".
- **Empty Set**: The empty set contains no elements (the intersection of the set of men and women is the empty set).
- **Cardinal Number**: The cardinal number of a set is the number of elements in a set (The set of primary colors has a cardinal number of 3). The cardinality of a set is denoted by "n".
- **Set Equality**: Two sets are equal if every element of one set is in the other set and the other set has no additional elements (The set of the letters in the word "god" is equal to the set of the letters in the work "dog").
- **Subset:** A subset is a set that only contains elements found in a larger set (The set of odd numbers from 1 to 10 is a subset of the set of numbers from 1 to 10).
- **Mutually Exclusive:** Sets are mutually exclusive if they do not share any of the same elements (the set of animals that are dogs and the set of animals that are cats are mutually exclusive).

Set Operations

- *Intersection:* The intersection of sets is a subset of the elements that the sets share (the set of letters in the word "work" and the word "stork" have in intersection of "ork").
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$
- *Union:* The union of sets is a set containing all of the elements from each set (the union of the set letters in the word "work" and the word "stork" is "stwork").
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$
- *Complement:* The complement of a set is all of the elements in the universal set, not in that set (the complement of the set of pages in Chapter 2 is the set of pages in the rest of the MATH 101 textbook, where the total number of pages in the book is the universal set).
$$A' = \{x \mid x \in U \text{ and } x \notin A\}$$
- *Finding Cardinal Numbers:*
- Union/Intersection:
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

o Complement
$$n(A) + n(A') = n(U)$$

De Morgan's Laws

- The complement of the union of sets is the intersection of the complements of each set (the letters not represented in the union of the sets of letters the words "bed" and "cat" is the same as the intersection of all the letters not in "bed" with all of the letters not in "cat").
$$(A \cup B)' = A' \cap B'$$
- The complement of the intersection of sets is the union of the complements of each set (the students not in the intersection of the classes MATH 101 and ENGL 102 is the same as the union of students not in MATH 101 and not in ENGL 102).
$$(A \cap B)' = A' \cup B'$$

Venn Diagrams

- Circles: Each circle in a Venn Diagram represents a set.
- Overlap of Circles: The overlap between circles is the intersection between those sets.
- The Box: The box represents the universal set. Everything inside the box that is not in a circle is the complement of all of the sets that are represented by circles.

Symbol Guide

Set Theory Symbols		
Term	Symbol	Use
Union	∪	$A \cup B$
Intersection	∩	$A \cap B$
Complement	'	A'
Subset	⊆	$A \subseteq B$
An Element of	∈	$a \in A$
Cardinality	n	$n(A)$
Universal Set	U	
Empty Set	∅	

Counting

The Fundamental Principle of Counting

- The total number of possible outcomes of a scenario where multiple decisions are made in no particular order is found by multiplying the total number of options for each decision.
- Note: It is usually wise to draw a blank for each decision category and put a multiplication sign between each. Then, fill each blank in with the number of options for that decision and multiply them.
- Example: What are the total number of possible pizzas created if the decision categories are crust, vegetables, and meat and there are three choices of crust (thin, traditional, deep dish), five choices of vegies (peppers, mushrooms, onions, spinach, olives), and four choices of meat (chicken, bacon, hamburger, sausage)?
  There are 3 categories with 3, 5, and 4 options, respectively. $\underline{3} \times \underline{5} \times \underline{4} = 60$. There are 60 possible different pizzas that can be made by the above options.

Tree Diagrams

- Tree diagrams represent the decision process visually. As with the above example, the order of the decisions will not affect the outcome; however, you need to make sure that each category has its own level in the diagram.
- Note: The final answer is the same as using the Fundamental Principle of Counting, but the tree diagram would have shown you exactly what each different pizza combination would be.

Factorials

- Factorials are a mathematical operation just like addition or multiplication. A **factorial** of a number n, called $n\ factorial$, and represented as $n!$ is the product of all positive integers from 1 to n.
$$n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$$
- Note: We define $0! = 1$ and do not use the factorial operation for negative numbers.
- Example: $6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$
- Note: Factorials get large very quickly and even the best calculators cannot compute past two digit factorials. If you find that your calculator gives you an error message when trying to calculate an answer, look for something that you can cancel the factorial with to make the values smaller. Just like 4 and 6 have a factor of 2 that can be cancelled when dividing, factorials share common factors that can be cancelled to make the operation simpler.
- Example: $\frac{6!}{4!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 5 \cdot 6 = 30$ Notice that the common factors on top and bottom cancel to leave only a pair of numbers to be multiplied.

Permutations

- The number of **permutations** (ordered arrangements) of r items picked from a group of n items without putting the picked items back into the group, represented by $_nP_r$, is

$$_nP_r = \frac{n!}{(n-r)!}$$

- Example: Select a first, second, and third prize for best costume at Regent's annual Halloween Party out of 105 attendees.

We know that we are looking for permutations because each prize placing is distinct.

$$_{105}P_3 = \frac{105!}{(105-3)!} = \frac{105 \cdot 104 \cdot 103 \cdot 102 \cdot 101 \cdots 3 \cdot 2 \cdot 1}{102 \cdot 101 \cdots 3 \cdot 2 \cdot 1} = 105 \cdot 104 \cdot 103 = 1,124,760$$

There are 1,124,760 possible ways in which three of the attendees can be picked to be first, second, and third prize. Notice how cancelling was used to turn large factorials into manageable multiplication.

Combinations

- The number of **combinations** (unordered arrangements) of r items picked from a group of n items without putting the picked items back in the group, represented by $_nC_r$, is

$$_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

- Example: Select 5 people to sit at a table of a university function out of 92 people who RSVPed to the event.

We know we are looking for combinations because the chairs at this table are not distinct in value.

$$_{92}C_5 = \frac{92!}{(92-5)! \cdot 5!} = \frac{92 \cdot 91 \cdot 90 \cdot 89 \cdot 88 \cdot 87 \cdot 86 \cdots 2 \cdot 1}{(87 \cdot 86 \cdot 85 \cdots 2 \cdot 1) \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{92 \cdot 91 \cdot 90 \cdot 89 \cdot 88}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 92 \cdot 91 \cdot 6 \cdot 89 \cdot 11 \cdot 1$$
$$= 49,177,128$$

There are 49,177,128 ways to put 5 distinct combinations of people at a table. Notice that every person at a table does not have to change to make a combination distinct.

- Note: If selecting the same number of items from the same size pool of options, there will be fewer combinations than permutations.

Symbol Guide

Counting Symbols		
Term	Symbol	Use
Factorial	!	$n!$
Permutation	$_nP_r$	ordered
Combination	$_nC_r$	unordered
Pool of items	n	
Selected items	r	